PHYS 461: Homework # 6

6.1 The numbers $1, 2, \ldots, n$ are arranged in random order.

The probability that the number 2 follows the number 1 is equal to

$$\frac{1}{n}$$

since all the possible outcomes:

$$2, 3, \ldots, n, \text{blank}$$

have the same probability.
The probability that the number 3 follows the sequence $(1, 2)$ is equal to

$$\frac{1}{n(n-1)}$$

since all the possible outcomes:

$$3, 4, \ldots, n, \text{blank}$$

have the same probability.
The probability that the sequence $1, 2, 3, 4$ appears in that order anywhere in the arrangement is, therefore, equal to

$$\frac{1}{n(n-1)(n-2)}.$$

Other solution Any permutation of the set

$$\{\{1, 2, 3, 4\}, 5, 6, \ldots, n\}$$

is a "favorable" case. Since this set contains $n-3$ elements, the probability that the sequence $1, 2, 3, 4$ appears in that order anywhere in the arrangement is, therefore, equal to

$$\frac{(n-3)\,!}{n\,!} = \frac{1}{n(n-1)(n-2)}.$$

6.2 The sequence $(c_1, c_2, \ldots, c_k)$ being of length k, the probability to obtain such a sequence if only k digits are drawn with replacement is 10^{-k}. Since the probability not to obtain it is $1 - 10^{-k}$, the probability not to obtain it after nk draws is $(1 - 10^{-k})^n$. When nk balls are successively drawn with replacement, the probability $p(k, n)$ to obtain the sequence of k digits $(c_1, c_2, \ldots, c_k)$ is, therefore, given by

$$p(k, n) = 1 - (1 - 10^{-k})^n.$$

The limit, when n tends to infinity of $p(k, n)$ is equal to 1. That is, *when nk balls are successively drawn with replacement, the probability to obtain the sequence of k digits $(c_1, c_2, \ldots, c_k)$ becomes a* **certain** *event in the limit $n \to \infty$.*

6.3 The number of different hands at bridge is

$$\binom{52}{13} \approx 6.35016 \times 10^{11}.$$

The 52 cards being partitioned between the 4 players, the number of different distributions is

$$\frac{52\,!}{(13\,!)^4} \approx 5.36447 \times 10^{28}.$$

The four aces may be ordered in $4\,! = 24$ different ways—each order represents one possibility of giving one ace to each player. The remaining 48 cards can be distributed in

$$\frac{48\,!}{(12\,!)^4}.$$

The probability that each player has an ace is, therefore,

$$24 \times \frac{\frac{48\,!}{(12\,!)^4}}{\frac{52\,!}{(13\,!)^4}} \approx 0.105498.$$

6.4 An urn contains b blue balls and r red balls. Let B_n denote the event: "the nth ball drawn from the urn is blue." We have

$$P(B_2) = P(B_1)P(B_2 \mid B_1) + P(\overline{B_1})P(B_2 \mid \overline{B_1}),$$

where

$$P(B_1) = \frac{b}{b+r}, \quad P(\overline{B_1}) = \frac{r}{b+r},$$
$$P(B_2 \mid B_1) = \frac{b-1}{b+r-1}, \quad P(B_2 \mid \overline{B_1}) = \frac{b}{b+r-1}.$$

Replacing in the expression of $P(B_2)$, we find

$$\begin{aligned} P(B_2) &= \frac{b}{b+r}\,\frac{b-1}{b+r-1} + \frac{r}{b+r}\,\frac{b}{b+r-1} \\ &= \frac{b(b+r-1)}{(b+r)(b+r-1)} \\ &= \frac{b}{b+r} \\ &= P(B_1). \end{aligned}$$

6.5 Initially, an urn contains only one red ball. We flip a coin. If the result is tail, a blue ball is added to the urn, if it is head, a ball is drawn from the urn.

Let B denote the event "we draw a red ball" and F_n the event "we obtain head for the first time at the nth flip of the coin." We have

$$P(F_n) = \frac{1}{2^n} \quad \text{and} \quad P(B \mid F_n) = \frac{1}{n},$$

since after having obtained tail $(n-1)$ consecutive times, the urn contains n balls. Hence

$$\begin{aligned} P(B) &= \sum_{n=1}^{\infty} P(F_n)P(B \mid F_n) \\ &= \sum_{n=1}^{\infty} \frac{1}{2^n}\,\frac{1}{n} \\ &= \log 2. \end{aligned}$$