PHYS 461: Homework # 9

9.1 Consider an ideal gas of N relativistic particles with no rest mass occupying a volume V. The energy of such a particle is $\varepsilon = cp$, where c is the velocity of light and p the norm $||\mathbf{p}||$ of the momentum $\mathbf{p}$ of a particle.
(i) Assuming that the gas obeys classical statistics, find the expressions of the pressure P and the internal energy U in terms of temperature T, volume V, and number of particles N.
(ii) Express the product PV in terms of the internal energy.

9.2 The number N of particles of a system $\mathcal{S}$ in contact with a reservoir of particles $\mathcal{R}$ is not constant. Its average number of particles is given by

$$\langle N \rangle = kT \frac{\partial \log Z}{\partial \mu},$$

where Z is the grand partition function and μ the chemical potential of the reservoir.
(i) Show that the variance of N, defined as $\langle N^2 \rangle - \langle N \rangle^2$, is given by

$$k^2T^2 \left(\frac{\partial^2 \log Z}{\partial \mu^2} - \left(\frac{\partial \log Z}{\partial \mu} \right)^2 \right).$$

(ii) Show that the above result can be written

$$\langle N^2 \rangle - \langle N \rangle^2 = kT \frac{\partial \langle N \rangle}{\partial \mu}.$$

(iii) Show that for an ideal gas in the classical regime, we have

$$\frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{1}{\langle N \rangle}.$$

9.3 The grand potential Ω of an ideal gas is given, either for bosons (upper signs) or fermions (lower signs), by

$$\Omega(T, V, \mu) = \pm \frac{AV}{\beta} \int_0^\infty \sqrt{\varepsilon} \log \left(1 \mp \mathrm{e}^{-\beta(\varepsilon - \mu)} \right) d\varepsilon,$$

where A is a constant and $\beta = 1/T$ (*i.e.*, $k = 1$).

(i) Show that we have

$$PV = VT^{5/2} f_1 \left(\frac{\mu}{T} \right),$$
$$S = VT^{3/2} f_2 \left(\frac{\mu}{T} \right),$$
$$\langle N \rangle = VT^{3/2} f_3 \left(\frac{\mu}{T} \right),$$

where f_1, f_2 and f_3 are three functions of the variable μ/T only. These functions are different for bosons and fermions.

(ii) Find the equations of the quasistatic adiabats in terms of, respectively, the pair of variables T and V, T and P, and V and P.

Hint: Show first that $S/\langle N \rangle$ is a function of μ/T only.