PHYS 461: Final Exam

1 From the relations

$$\frac{\chi_T}{\chi_S} = \frac{C_P}{C_V}, \quad \text{and} \quad C_P - C_V = \frac{TV\alpha^2}{\chi_T},$$

where C_V is the heat capacity at constant volume, it follows that

$$\begin{aligned}
\chi_T - \chi_S &= \chi_T \left(1 - \frac{\chi_S}{\chi_T}\right) \\
&= \chi_T \left(1 - \frac{C_V}{C_P}\right) \\
&= \chi_T \left(\frac{C_P - C_V}{C_P}\right) \\
&= \frac{TV\alpha^2}{C_P}.
\end{aligned}$$

2 (i) Since n fish are marked among N,

- $\binom{N}{n}$ is the number of different ways to choose the n fish to be marked among N.
- $\binom{n}{k}$ is the number of different ways to choose k marked fish among n.
- $\binom{N-n}{n-k}$ is the number of different ways to choose the $n-k$ fish among the $N-n$ non marked ones.

Hence,

$$p(N,n,k) = \frac{\binom{n}{k}\binom{N-n}{n-k}}{\binom{N}{n}}.$$

$2n-k$ being the number of different fish caught, this result makes sense only if $N > 2n-k$.

Consider

$$\frac{p(N,n,k)}{p(N-1,n,k)} = \frac{\binom{n}{k}\binom{N-n}{n-k}\binom{N-1}{n}}{\binom{n}{k}\binom{N-1-n}{n-k}\binom{N}{n}}$$
$$= \frac{(N-n)^2}{N(N-2n+k)}.$$

From

$$\frac{(N-n)^2}{N(N-2n+k)} > 1 \quad \Rightarrow \quad n^2 > kN$$
$$\frac{(N-n)^2}{N(N-2n+k)} < 1 \quad \Rightarrow \quad n^2 < kN$$

Hence, as a function of N, $p(N,n,k)$ increases for $N < n^2/k$, and decreases for $N > n^2/k$. The probability $p(N,n,k)$ is, therefore, maximum when N is the largest integer less than or equal to n^2/k.

That is, the *maximum likelihood estimate of* N is given by

$$N_{\max} = \left\lfloor \frac{n^2}{k} \right\rfloor,$$

where $\lfloor x \rfloor$ is largest integer less than or equal to x.

If $n = 1000$ and $k = 100$, the maximum likelihood estimate of the number of fish is $N_{\max} = 10000$.

3 (i) The partition function $Z(N, \beta, B)$ of a N noninteracting spins $\frac{1}{2}$ in an external magnetic field B directed along the z-axis is given by the sum over all the values of the z-components of all the spins $\mathbf{S}^i$. Since

$$\exp\left(\beta g \mu B \sum_{i=1}^{N} S_z^i\right) = \prod_{i+1}^{N} \exp(\beta g \mu B S_z^i),$$

the partition function is given by

$$\begin{aligned} Z(\beta, B, N) &= \left(\exp(\tfrac{1}{2}\beta g \mu B) + \exp(\tfrac{1}{2}\beta g \mu B)\right)^N \\ &= \left(2\cosh(\tfrac{1}{2}\beta g \mu B)\right)^N . \end{aligned}$$

(ii) The Gibbs potential is given by

$$\begin{aligned} G(\beta, B, N) &= -\frac{1}{\beta} \log Z(\beta, B, N) \\ &= -\frac{N}{\beta} \log\left(2\cosh(\tfrac{1}{2}\beta g \mu B)\right). \end{aligned}$$

(iii) The entropy is given by

$$S(\beta, B, N) = -\frac{\partial G}{\partial \beta}\,\frac{d\beta}{dT} = k\beta^2\frac{\partial G}{\partial \beta}.$$

Hence

$$\begin{aligned} S(\beta, B, N) = {} & kN\log\left(2\cosh(\tfrac{1}{2}\beta g\mu B)\right) \\ & - \tfrac{1}{2}\,kN\beta g\mu B\tanh(\tfrac{1}{2}\beta g\mu B). \end{aligned}$$

(iv) The specific heat per spin at constant field is defined as

$$c_B(\beta, B) = \frac{T}{N}\,\frac{\partial S}{\partial T} = -\frac{\beta}{N}\,\frac{\partial S}{\partial \beta},$$

that is,

$$c_B(\beta, B) = \frac{k\beta^2 g^2\mu^2 B^2}{4\cosh^2(\frac{1}{2}\beta g\mu B)}.$$

This quantity is obviously positive.

(v) When, for a given value of the applied field B, β tends to infinity, the hyperbolic cosine grows much faster than β^2, thus $c_B(\beta, B)$ tends to zero. Since $\beta \to \infty$ means $T \to 0$, this result agrees with the third principle of thermodynamics.