Configurational Equilibrium of Cracks Affected by Surface Stress and Crack-Tip Point Load

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Configurational Equilibrium

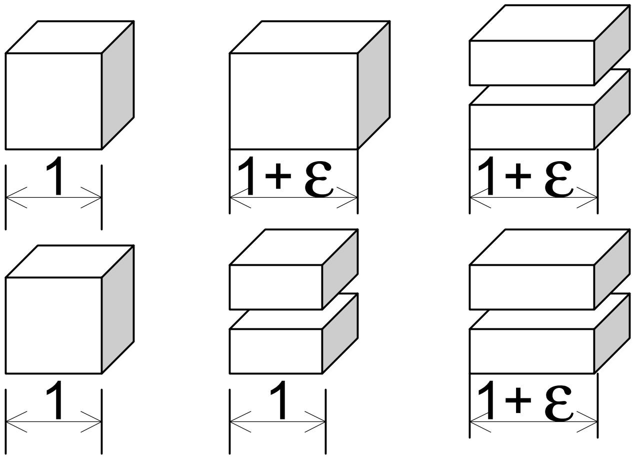
$$\frac{\partial(\text{Potential Energy})}{\partial(\text{Configuration})} = 0$$

Effect of Surface Stress and Crack-Tip Load

Surface Stress

Stretch Then Split

Split Then Stretch



Stretch unit cube along x, then split

$$W(\varepsilon) + 2\Gamma(\varepsilon) = \text{Work A}$$

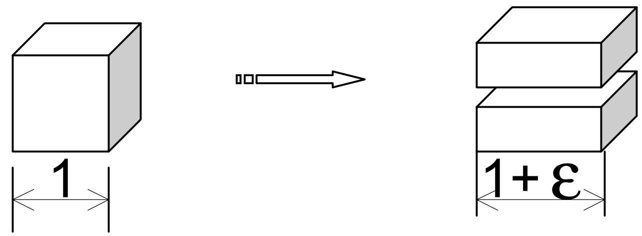
Split unit cube, then stretch along x

$$2\Gamma(0) + \left(W(\varepsilon) + 2\Sigma\varepsilon\right) = \text{Work B}$$

Work A = Work B

$$\Sigma = \partial\Gamma / \partial\varepsilon$$

Referential and Spatial Forms



$\Gamma(\varepsilon)$: **Surface Energy per Unit Referential Area**

$\gamma(\varepsilon)$: **Surface Energy per Unit Spatial Area**

$$\gamma(\varepsilon) = \frac{1}{1+\varepsilon}\Gamma(\varepsilon) = (1-\varepsilon)\Gamma(\varepsilon), \quad \Sigma = \frac{\partial \Gamma}{\partial \varepsilon} = \gamma(\varepsilon) + \frac{\partial \gamma}{\partial \varepsilon}$$

Linear Density: $\Gamma(\varepsilon) = \Gamma_o + \Sigma_o \varepsilon, \quad \gamma(\varepsilon) = \Gamma_o + (\Sigma_o - \Gamma_o)\varepsilon$

Γ_o: **Surface Tension Coefficient**

Σ_o: **Surface Stress Coefficient**

Surface Stress Coefficient

Behaves like a pre-stressed membrane that is perfectly fitted on the bounding surface of a bulk material body.

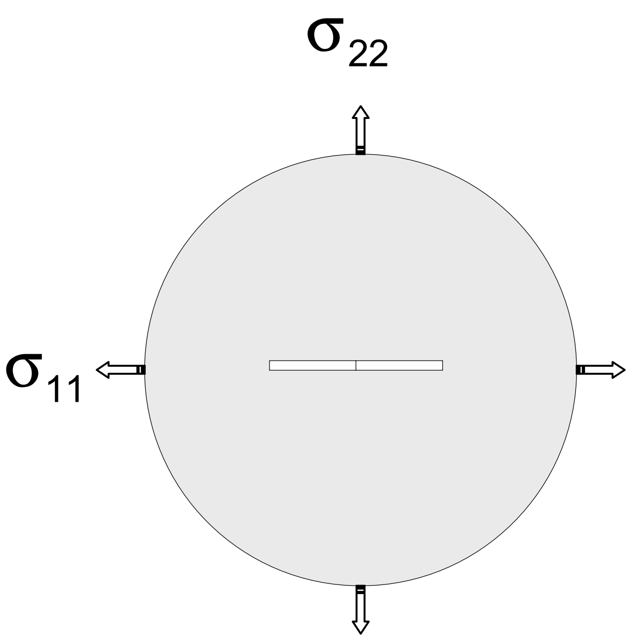
Interacts with the deformation of the bulk material through the curvature of the surface.

Becomes increasingly important at reduced scales.

Effect on linear and curvilinear cracks.

Configurational Energy

Griffith Crack (1921 & 1924)



$R \to \infty$

Configurational Energy:

$$\Pi = \Pi_0 + \Delta\Pi$$

Stress Intensity Factor:

$$K_1 = \sigma_{22}\sqrt{\pi a}$$

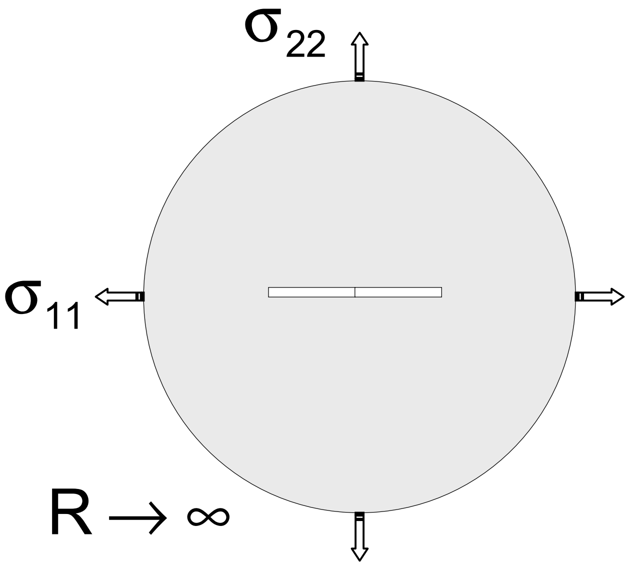
For Configurational Equilibrium:

$$\Delta\Pi = -\frac{\kappa+1}{8\mu}\pi a^2\sigma_{22}^2 + 4\Gamma_0 a$$

$$\frac{\partial\Delta\Pi}{2\partial a} = 0 \Rightarrow K_1^2 = K_{1C}^2 \equiv \frac{16\mu\Gamma_0}{\kappa+1}$$

Configurational Energy

Griffith Crack with Surface Stress Σ_0



Configurational Energy:

$$\Pi = \Pi_0 + \Delta\Pi$$

Stress Intensity Factor:

$$K_1 = \sigma_{22}\sqrt{\pi a} \quad \text{(same)}$$

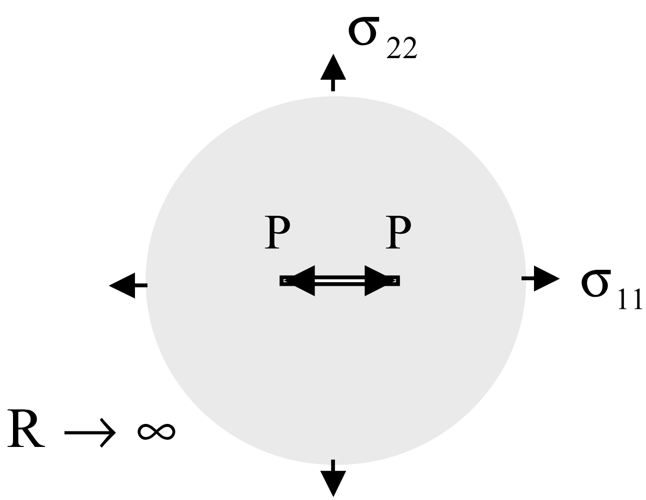
For Configurational Equilibrium:

$$\Delta\Pi = -\frac{\kappa+1}{8\mu}\pi a^2\sigma_{22}^2 + 4\Gamma_0 a - \frac{\kappa+1}{8\mu}4a\Sigma_0(\sigma_{22}-\sigma_{11})$$

$$\frac{\partial\Delta\Pi}{2\partial a} = 0 \quad \Rightarrow \quad K_1^2 + \frac{2\Sigma_0}{\sqrt{\pi a}}\left(1-\frac{\sigma_{11}}{\sigma_{22}}\right)K_1 - K_{1C}^2 = 0$$

Configurational Energy

Griffith Crack with Surface Stress Σ_0 and Crack-Tip Point Load P



Configurational Energy:

$$\Pi = \Pi_0 + \Delta\Pi$$

Stress Intensity Factor:

$$K_1 = \sigma_{22}\sqrt{\pi a} \quad \text{(still the same)}$$

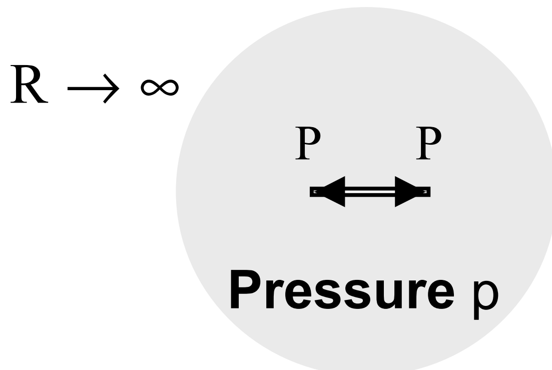
For Configurational Equilibrium:

$$\Delta\Pi = -\frac{\kappa+1}{8\mu}\pi a^2\sigma_{22}^2 + (4\Gamma_0 - 2P)a - \frac{\kappa+1}{8\mu}(4\Sigma_0 - 2P)(\sigma_{22} - \sigma_{11})a$$

$$\frac{\partial\Delta\Pi}{2\partial a} = 0 \quad \Rightarrow \quad K_1^2 + \frac{2\Sigma_0 - P}{\sqrt{\pi a}}\left(1 - \frac{\sigma_{11}}{\sigma_{22}}\right)K_1 - \left(1 - \frac{P}{2\Gamma_0}\right)K_{1C}^2 = 0$$

Configurational Energy

Surface Stress, Internal Pressure p,and Crack-Tip Load P



Configurational Energy:

$$\Pi = \Pi_0 + \Delta\Pi$$

Stress Intensity Factor:

$$K_1 = p\sqrt{\pi a}$$

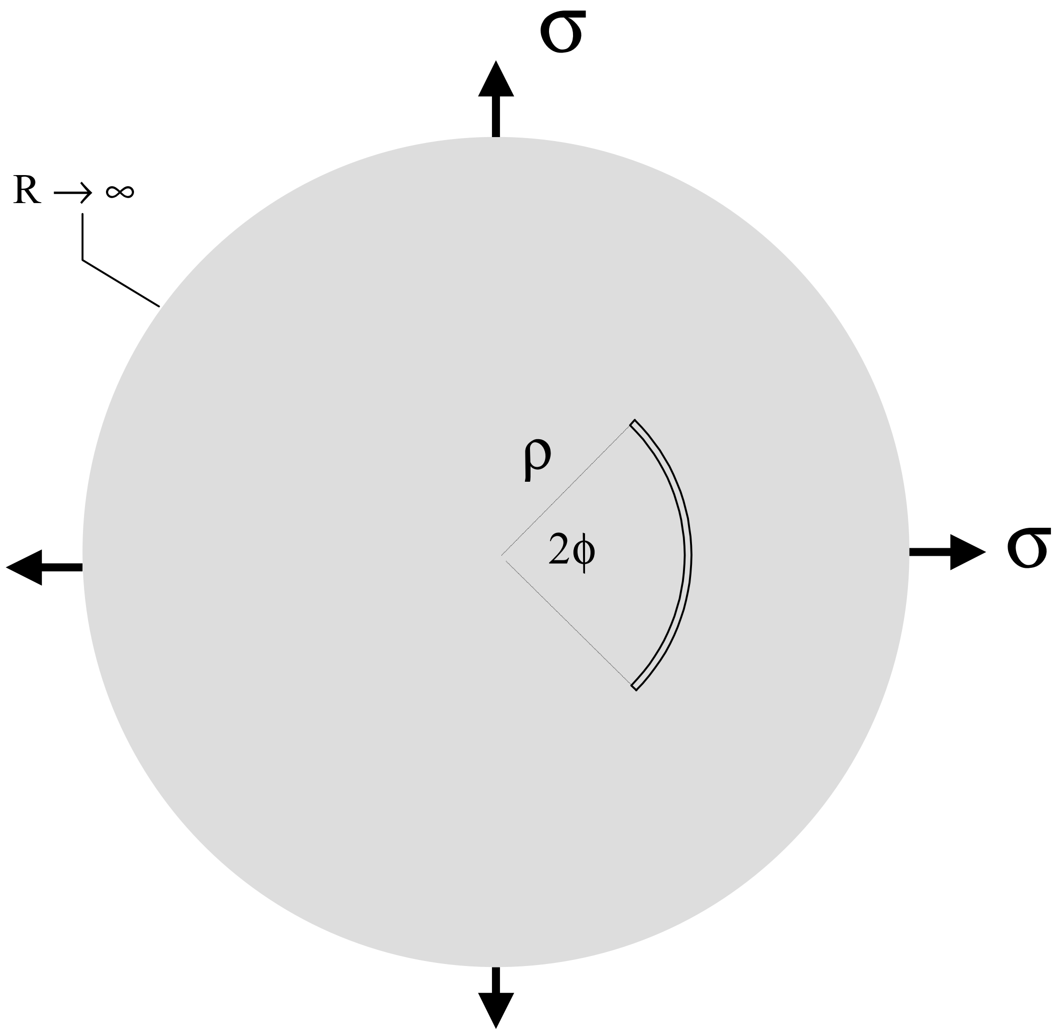
For Configurational Equilibrium:

$$\Delta\Pi = -\frac{\kappa+1}{8\mu}\pi a^2 p^2 + (4\Gamma_0 - 2P)a + \frac{\kappa-1}{2\mu}(2\Sigma_0 - P)pa$$

$$\frac{\partial\Delta\Pi}{2\partial a} = 0 \quad \Rightarrow \quad K_1^2 + \frac{(2\Sigma_0 - P)}{\sqrt{\pi a}}\frac{2(\kappa-1)}{(\kappa+1)}K_1 - \left(1 - \frac{P}{2\Gamma_0}\right)K_{1C}^2 = 0$$

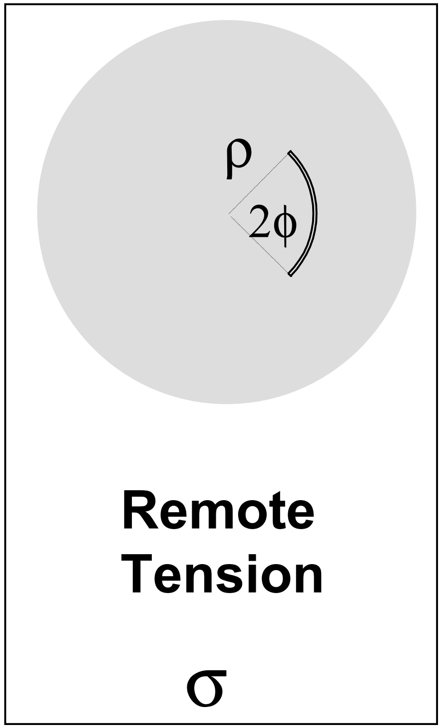
Configurational Energy

Circular-Arc Cracks, Remote Tension and Surface Stress



Configurational Energy

Circular-Arc Cracks, Remote Tension and Surface Stress



Remote Tension:

$$K^{(o)2} \equiv K_I^{(o)2} + K_{II}^{(o)2} = \sigma^2 \pi\rho \sin\phi \Big/ \left(1 + \sin^2\frac{\phi}{2}\right)^2$$

Remote Tension and Surface Stress:

$$K^2 \equiv K_I^2 + K_{II}^2 = \left(\sigma - \frac{\Sigma_0 \phi}{\pi\rho}\right)^2 \pi\rho \sin\phi \Big/ \left(1 + \sin^2\frac{\phi}{2}\right)^2$$

For Configurational Equilibrium:

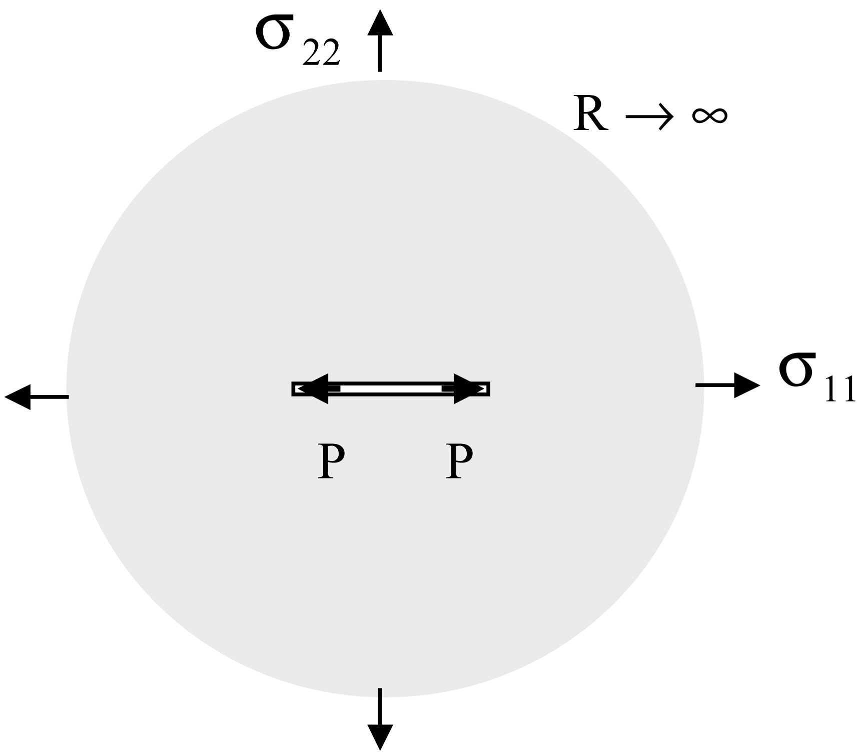
$$\Delta\Pi = 4\rho\phi\Gamma_0 - \frac{(\kappa+1)}{2\mu}\frac{(1-\cos\phi)}{(3-\cos\phi)}\left(\sqrt{\pi}\sigma\rho - \frac{\Sigma_0\phi}{\sqrt{\pi}}\right)^2$$

$$\frac{\partial \Delta\Pi}{2\rho\partial\phi} = 0 \quad \Rightarrow \quad K^2 - \left[4\Sigma_0 \sin^2\frac{\phi}{2} \Big/ \sqrt{\pi\rho\sin\phi}\right] K - K_{IC}^2 = 0$$

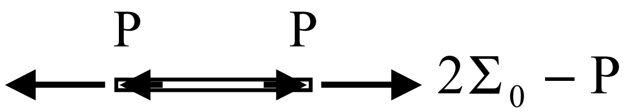
Elasticity Solution

Remote Loading, Surface Stress and Crack-Tip Load

The Bulk-Surface System



Free-Body Of The "Surface"



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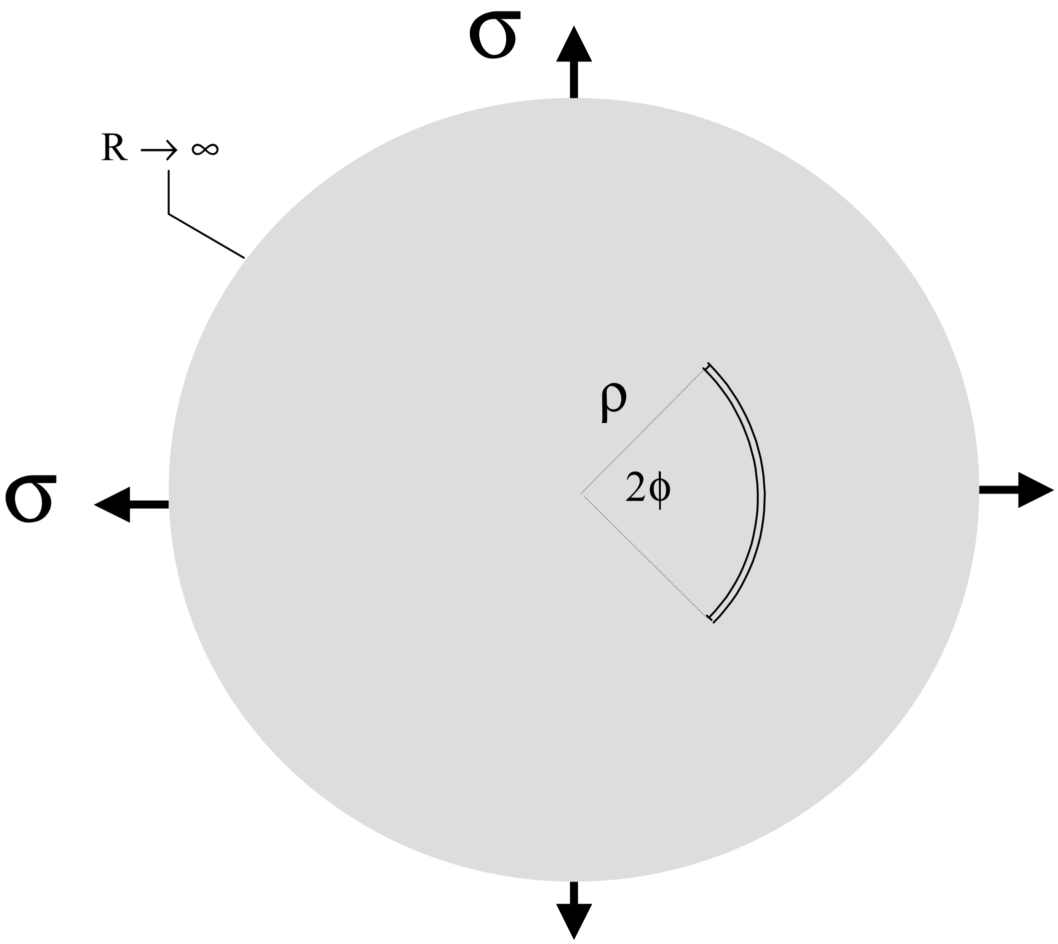
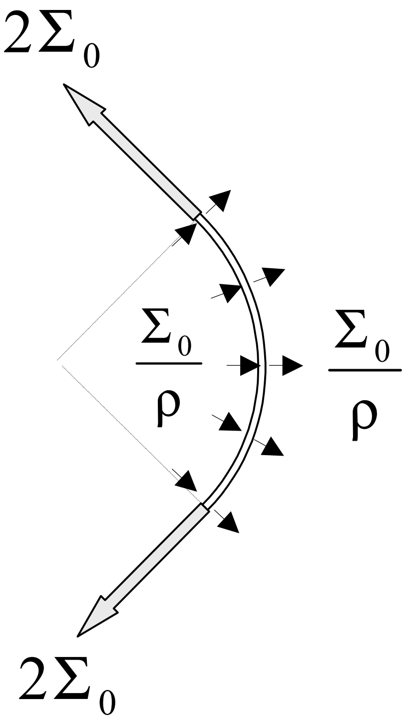
Elasticity Solution

Circular-Arc Crack,Remote Loading and Surface Stress



The Bulk-Surface System



Free-Body Of The "Surface"

Elasticity Solution - Griffith Crack

Remote Loading, Surface Stress and Crack-Tip Load

$$\left(u_\alpha, \varepsilon_{\alpha\beta}, \tau_{\alpha\beta}\right) = \left(u_\alpha^{(o)}, \varepsilon_{\alpha\beta}^{(o)}, \tau_{\alpha\beta}^{(o)}\right) + \left(u_\alpha^{(s)}, \varepsilon_{\alpha\beta}^{(s)}, \tau_{\alpha\beta}^{(s)}\right)$$

Total Load Surface Stress

Strain Energies:

$$U_e \equiv \int W\left(\varepsilon_{\alpha\beta}\right) dA = \frac{R}{2}\int_0^{2\pi}\left[\sigma_{rr}(\theta)u_r(R,\theta) + \sigma_{r\theta}(\theta)u_\theta(R,\theta)\right]d\theta$$
$$+\frac{1}{2}\left(P - 2\Sigma_0\right)\left[u_1(a,0) - u_1(-a,0)\right]$$

$$U_e^{(o)} \equiv \int W\left(\varepsilon_{\alpha\beta}^{(o)}\right) dA = \frac{R}{2}\int_0^{2\pi}\left[\sigma_{rr}(\theta)u_r^{(o)}(R,\theta) + \sigma_{r\theta}(\theta)u_\theta^{(o)}(R,\theta)\right]d\theta$$

$$U_e^{(s)} \equiv \int W\left(\varepsilon_{\alpha\beta}^{(s)}\right) dA = \frac{1}{2}\left(P - 2\Sigma_0\right)\left[u_1^{(s)}(a,0) - u_1^{(s)}(-a,0)\right]$$

Total Elastic Energy

$$U_e = \frac{R}{2}\int_0^{2\pi}\left[\sigma_{rr}(\theta)u_r^{(s)}(R,\theta) + \sigma_{r\theta}(\theta)u_\theta^{(s)}(R,\theta)\right]d\theta$$

$$+\frac{1}{2}(P - 2\Sigma_0)\left[u_1^{(o)}(a,0) - u_1^{(o)}(-a,0)\right]$$

$$+U_e^{(o)} + U_e^{(s)}$$

$$= (P - 2\Sigma_0)\left[u_1^{(o)}(a,0) - u_1^{(o)}(-a,0)\right] + U_e^{(o)} + U_e^{(s)}$$

Total Elastic Strain Energy:

$$U_e = U_e^{(o)} + U_e^{(s)} + U_e^{(os)}$$

Interaction Energy:

$$U_e^{(os)} = (P - 2\Sigma_0)\left[u_1^{(o)}(a,0) - u_1^{(o)}(-a,0)\right]$$

Energy of the Bulk-Surface System

Elastic Energy of the Bulk: $U_e = U_e^{(o)} + U_e^{(s)} + U_e^{(os)}$

Surface Energy:

$$U_s = 2\int_{-a}^{+a} \Gamma(\varepsilon) dx_1 = 4a\Gamma_0 + 2\Sigma_0 \left[u_1(a,0) - u_1(-a,0) \right]$$

Potential of External Loads:

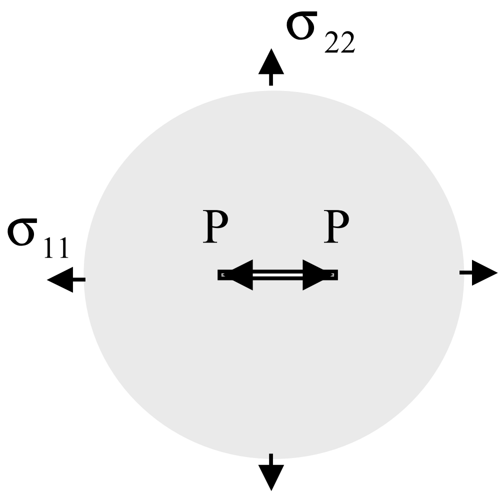
$$V = R\int_0^{2\pi} \left[\sigma_{rr}(\theta) u_r(R,\theta) + \sigma_{r\theta}(\theta) u_\theta(R,\theta) \right] d\theta$$
$$+ P\left[2a + u_1(a,0) - u_1(-a,0) \right]$$

Total Potential Energy:

$$\Pi = U_s + U_e - V = -U_e^{(o)} - U_e^{(s)} - U_e^{(os)} + (4\Gamma_0 - 2P)a$$

Final Results For σ_{11}, σ_{22}, P, Σ_0 and $R \to \infty$

$$U_e^{(o)} = \frac{\pi R^2}{16\mu}\left[(\kappa-1)(\sigma_{11}+\sigma_{22})^2 + 2(\sigma_{11}-\sigma_{22})^2\right] + \frac{\kappa+1}{8\mu}\pi a^2\sigma_{22}^2 + \cdots$$



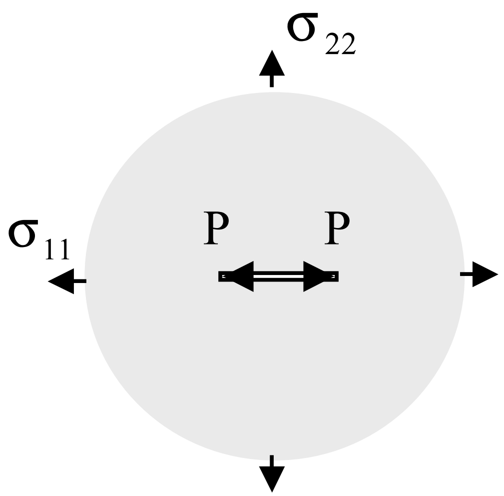
$$U_e^{(s)} = -\frac{(\kappa+1)\Sigma_0^2}{2\pi\mu}\ln\left|\frac{z-a}{z+a}\right| \quad z \to a$$

$$U_e^{(os)} = \frac{(P-2\Sigma_0)(\kappa+1)a}{4\mu}(\sigma_{11}-\sigma_{22})$$

$$\Delta\Pi = -\frac{\kappa+1}{8\mu}\pi a^2\sigma_{22}^2 + (4\Gamma_0 - 2P)a - \frac{\kappa+1}{8\mu}(4\Sigma_0 - 2P)(\sigma_{22}-\sigma_{11})a$$

$$\frac{\partial\Delta\Pi}{2\partial a} = 0 \quad \Rightarrow \quad K_1^2 + \frac{2\Sigma_0 - P}{\sqrt{\pi a}}\left(1-\frac{\sigma_{11}}{\sigma_{22}}\right)K_1 - \left(1-\frac{P}{2\Gamma_0}\right)K_{1C}^2 = 0$$

The Role of P

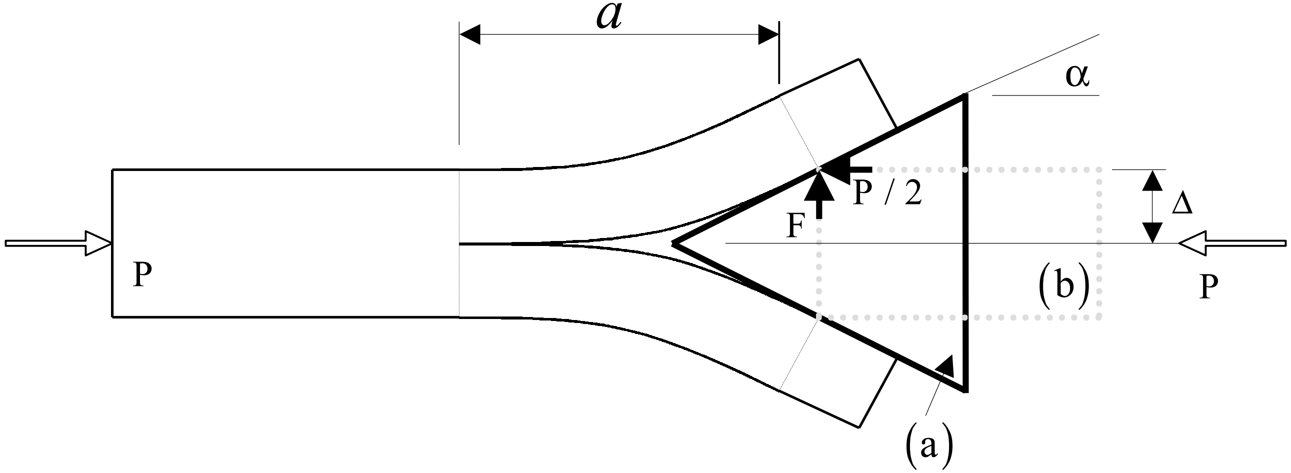


Let $\sigma_{11} = \sigma_{22} = 0$

Then $\Delta\Pi = (4\Gamma_0 - 2P)a$ **and**

$$\frac{\partial \Delta\Pi}{2\partial a} = 0 \quad \Rightarrow \quad P = 2\Gamma_0$$

Splitting A Double-Cantilever Beam



(a) A Smooth Wedge of Angle 2α

$$\Pi_\alpha = 2\left[\frac{2EI\alpha^2}{3a} + \Gamma_0 a\right], \qquad \frac{\partial \Pi_\alpha}{\partial a} = 0 \quad \Rightarrow \quad P = 6\Gamma_0$$

(b) A Smooth Edge of Thickness 2Δ

$$\Pi_\Delta = 2\left[\frac{3EI\Delta^2}{2a^3} + \Gamma_0 a\right], \qquad \frac{\partial \Pi_\Delta}{\partial a} = 0 \quad \Rightarrow \quad P = 2\Gamma_0$$