

Dynamic Pricing and Inventory Control: Uncertainty and Competition

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In this paper, we study a make-to-stock manufacturing system where two firms compete through dynamic pricing and inventory control. Our goal is to address competition (in particular a duopoly setting) together with the presence of demand uncertainty. We consider a dynamic setting where multiple products share production capacity. We introduce a demand-based fluid model where the demand is a linear function of the price of the supplier and of her competitor, the inventory and production costs are quadratic, and all coefficients are time dependent. A key part of the model is that no backorders are allowed and the strategy of a supplier depends on her competitor's strategy. First, we reformulate the robust problem as a fluid model of similar form to the deterministic one and show existence of a Nash equilibrium in continuous time. We then discuss issues of uniqueness and address how to compute a particular Nash equilibrium, i.e., the normalized Nash equilibrium.

Subject classifications: game theory; optimization under uncertainty; robust optimizations; normalized Nash equilibrium; dynamic pricing.

Area of review: Optimization.

History: Received January 2007; revisions received February 2008, June 2008, August 2008, January 2009; accepted January 2009. Published online in *Articles in Advance*.

1. Introduction

1.1. Motivation

Continuous-time optimal control models (sometimes also referred as fluid models) provide a powerful tool for understanding the behavior of systems where the dynamic aspect plays an important role. In this paper, we use a continuous-time optimal control model in the setting of dynamic pricing and inventory management. Our goal is to efficiently take into account the impact of decisions on the state of the system over a time horizon and model the evolution of the inventory levels. One of the critical factors to determine the profitability of a firm is its pricing and production strategy. Therefore, companies' ability to survive in this very competitive environment depends on the development of efficient pricing models. Inventory control can also have a major impact on profits. In a nonmonopolistic setting, the decisions taken by a firm may affect other firms both in terms of their profits and their feasible strategies. Such problems have been studied in the literature in economics, revenue management, and supply chain management. Because multiple suppliers are simultaneously maximizing their own revenue without cooperating, we focus on finding a Nash equilibrium, i.e., a set of strategies such that no supplier has any incentive to unilaterally deviate. In this paper, we formulate the problem as a differential game.

The demand each supplier faces for each product she is selling is uncertain. Nevertheless, in practice, it is very difficult to forecast demand accurately. This is especially true in a competitive setting because each supplier needs to forecast how demand is affected not only by its own prices but also by its competitors' prices. Economists have used a variety of demand models which have been applied in the revenue management literature (see, for example, Vives 1999 and the survey articles by Chan et al. 2004, Elmaghraby and Keskinocak 2003). Nevertheless, the validity of these models relies on the ability to determine the exact values of the price parameters (also referred to as price sensitivities). A common approach to deal with this uncertainty on the demand is to assume a probability distribution on the parameters. However, suppliers may find it difficult in practice to determine such probability distributions. Therefore, it may be more realistic to assume a range of variation on the parameters rather than a probability distribution. This idea motivates a robust optimization approach. The main idea is to find the best strategy given some uncertainty on the data.

The overall goal of this research is to introduce and study a robust optimization model and its application to dynamic pricing and inventory control in a duopoly setting. In Adida and Perakis (2006) we considered a monopolistic setting with demand uncertainty and an additive demand model. In this paper, we incorporate the element of competition

(using ideas from quasivariational inequalities) combined with demand uncertainty (using ideas from robust optimization). Furthermore, in this paper we assume a more general model of demand where additive and multiplicative models are special cases.

1.2. Some Related Literature and Contributions

There is a huge literature on inventory control as well as pricing and revenue management. For example, the book by Talluri and van Ryzin (2004) provides an overview of the revenue management and pricing literature. We use ideas from the fast growing field of robust optimization. For example, we use ideas from Ben-Tal and Nemirovski (1998) and Bertsimas and Sim (2004). We refer the reader to Adida and Perakis (2006) for additional references on related literature on dynamic pricing, inventory management, fluid models, demand uncertainty, and robust optimization.

Literature on oligopolistic competition in the field of pricing and revenue management is emerging fast in recent years. The book by Vives (1999) presents a number of pricing models in an oligopoly market. A wide range of applications of pricing can be found in the literature in economics, marketing, and management science (see, for example, Dockner et al. 2000 and the references therein). For a survey of applications of differential games to management science, see Jørgensen et al. (1999). In many cases in these applications, the equilibrium point is obtained by solving the differential form of the Kuhn-Tucker optimality conditions. Nevertheless, it is not always the case that solving these conditions is tractable.

Differential games are useful to model dynamic systems of conflict and cooperation where decisions are made over a time horizon. Başar presents some theoretic results in Başar and Olsder (1995) for noncooperative dynamic games. Mosco (1976) studies topological and order methods to solve quasivariational inequalities with implicit constraints and connects these problems with applications to differential games. Quasivariational inequalities in finite dimension have been studied by several authors such as Pang and Fukushima (2005). Cavazzuti et al. (2002) introduce some relationships between Nash equilibria, variational equilibria, and dynamic equilibria for noncooperative games without assuming that the dimension is finite. Cubiotti (1997) proves the existence of a solution for generalized quasivariational inequalities in infinite-dimensional normed spaces under some conditions.

Game-theoretic problems have motivated a lot of research. The degree of interdependency between players' actions impacts directly the difficulty of finding an equilibrium. The majority of literature assumes that, although players' payoffs depend on the other players' strategies, their strategy sets are independent. However, Arrow and Debreu (1954) point out the need for a generalization to an "abstract economy," in which the choice of an action by one agent affects both the payoff and the domain of actions of other

agents. Rosen (1965) studied finite-dimensional n -person nonzero sum games and considered games in which the constraints as well as the payoff function depend on the strategy of every player. Rosen referred to these games as coupled constraint games. In the literature, this class of games has also been referred to as social equilibria games, pseudo-Nash equilibria games, and generalized Nash equilibria games. Rosen (1965) introduced the notion of normalized Nash equilibrium as a particular Nash equilibrium.

Relaxation algorithms provide a powerful tool for finding a Nash equilibrium when the problem is not tractable enough to solve the necessary conditions. These iterative algorithms rely on averaging the current solution iterate with the solution of the *best-response* problem each player solves. Uryas'ev and Rubinstein (1994) and Başar and Olsder (1995) study the convergence of such algorithms in finite dimensions for finding the equilibria of a noncooperative game for some payoff functions on a closed, compact subset of \mathbb{R}^m . Berridge and Krawczyk (1997) apply these techniques to games with nonlinear payoff functions and coupled constraints arising in economics.

The contributions of this paper are the following.

1. We introduce robust optimization to a fluid model in a duopoly setting. To the best of our knowledge, this is the first paper combining competition together with demand uncertainty and the dynamic nature of the problem.

2. We formulate the deterministic problem as a coupled constraint (both the payoff function and the feasible strategy sets depend on the competitor's policy) differential game and prove the existence of a Nash equilibrium in continuous time, i.e., in an infinite-dimensional setting.

3. We provide a deterministic problem that is equivalent to the robust formulation and show that the existence result holds for this problem as well. We reformulate the robust problem into an equivalent robust counterpart that is similar to the nominal problem.

4. We discretize time and introduce a robust optimization formulation for the discrete-time model in a duopoly and show existence and uniqueness of a particular Nash equilibrium, namely, the normalized Nash equilibrium.

5. To compute this equilibrium, we introduce a relaxation algorithm and prove its convergence to the unique normalized Nash equilibrium.

In §2, we describe the nominal problem and introduce some basic properties. In §3, we present a model of demand uncertainty and study a robust formulation. In particular, we write the robust counterpart problem as a deterministic problem and show existence of a Nash equilibrium in continuous time for the robust formulation. We also provide an equivalent tractable formulation. In §4, we present a discretized robust formulation and describe the notion of a normalized Nash equilibrium. In §5, we show existence and uniqueness of this equilibrium. In §6, we introduce a relaxation algorithm and prove its convergence. Finally, in §7, we discuss some insights through a numerical example.

2. Deterministic Model

Superscript k ($k = A, B$) denotes supplier k . Superscript $-k$ denotes supplier k 's competitor (i.e., if $k = A$, then $-k = B$; and if $k = B$, then $-k = A$).

2.1. Formulation of the Deterministic Problem

We first present the problem of dynamic pricing and inventory control in a duopoly setting. To describe the basic problem formulation and discuss the associated assumptions, first in this subsection we consider the problem where all parameters are assumed to be known with no uncertainty. In this paper, two suppliers denoted by $k = A, B$ produce N differentiated, nonperishable products denoted by $i = 1, \dots, N$ over a finite time horizon $[0, T]$. A common production capacity among all products may vary over time for each supplier. Backorders are not allowed and the production costs and inventory costs are assumed to be quadratic, with time-dependent coefficients. The demand for a product of a given supplier is modeled as a linear function of the supplier's price and of the competitor's price for the same product, with time-dependent parameters:

$$d_i^k(t) = \alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t) \quad \forall i, t, k = A, B,$$

where $\beta_i^{k,l}(\cdot) > 0$ denotes the demand sensitivity of supplier k for product i , with respect to the price of product i applied by supplier l . Note that a supplier's demand is a strictly decreasing function of her own price and an increasing function of her competitor's price for the same product.

ASSUMPTION 1. Assume that the following inequalities hold $\forall i, t$: $0 \leq \beta_i^{k,-k}(t) < \beta_i^{k,k}(t)$, $k = A, B$ and $0 \leq \beta_i^{k,-k}(t) < \beta_i^{-k,-k}(t)$, $k = A, B$.

The first condition states that the demand observed by a given supplier should be more sensitive to that supplier's price rather than to her competitor's price of the same product. The second condition means that the price chosen by a given supplier affects her own demand more than it affects her competitor's demand. Both are standard assumptions in economics (see Vives 1999).

The dynamic dimension of the problem is taken into account by introducing a fluid model for the inventory level, in which the instantaneous change of inventory is given by the difference between the production rate and the demand rate. In other words, the inventory level at time t is given by adding to the initial inventory level the cumulative production up to time t , and subtracting the cumulative demand up to time t . In this model, each supplier wants to maximize her net profits over the time horizon by determining at time $t = 0$ the optimal pricing and production policy for the entire time horizon, subject to some constraints. The fact the suppliers' demand depends on the competitor's pricing strategy places us

within a game-theoretic framework. More details on motivation on the model can be found in the online appendix. An electronic companion to this paper is available as part of the online version that can be found at <http://orjournal.informs.org/>.

The best-response problem faced by supplier k , given price $p_i^{-k}(\cdot)$ for product i for the competitor, may be formulated as follows:

$$\begin{aligned} \max_{p_i^k, u_i^k} \quad & \int_0^T \sum_{i=1}^N (p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) \\ & - \gamma_i^k(t)(u_i^k(t))^2 - h_i^k(t)(I_i^k(t))^2) dt \\ \text{s.t.} \quad & \dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t) \\ & \forall t \in [0, T], i = 1, \dots, N, \quad (1) \\ & \sum_{i=1}^N u_i^k(t) \leq K^k(t) \quad \forall t \in [0, T], \quad (2) \\ & p_i^k(t) \leq \frac{\alpha_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)}{\beta_i^{k,k}(t)} \\ & \forall t \in [0, T], i = 1, \dots, N, \quad (3) \\ & I_i^k(t) \geq 0 \quad \forall t \in [0, T], i = 1, \dots, N, \quad (4) \\ & u_i^k(t), p_i^k(t) \geq 0 \quad \forall t \in [0, T], i = 1, \dots, N, \\ & I_i^k(0) = I_i^{k0}, \quad i = 1, \dots, N. \end{aligned}$$

In this formulation, the objective function describes the profit by adding over all products and over the time horizon the price multiplied by the demand, and subtracting all costs, that is, the quadratic production and inventory costs. Fluid Equation (1), along with the initial condition, determines the inventory level by ensuring that the change of inventory level is the difference between the production rate and the demand rate. Constraint (4) ensures that there are no backorders. The upper bound on the price (3) comes from the fact that the demand rate should remain nonnegative. Finally, constraint (2) is the capacity constraint on the production.

We notice that not only the objective function, but also the set of feasible strategies for a player, depend on the pricing strategy of the other player via the no backorder constraint (because the inventory level for one supplier depends on the other supplier's price) and the upper bound on the price. We assume that the competitors make their choices simultaneously. As a result, we have to study a deterministic, noncooperative, coupled constrained, nonzero sum, two-person differential game of prespecified duration $[0, T]$.

REMARK. Notice that combining the feasibility conditions

$$\begin{aligned} p_i^A(t) & \leq \frac{\alpha_i^A(t)}{\beta_i^{A,A}(t)} + \frac{\beta_i^{A,B}(t)}{\beta_i^{A,A}(t)} p_i^B(t) \quad \text{and} \\ p_i^B(t) & \leq \frac{\alpha_i^B(t)}{\beta_i^{B,B}(t)} + \frac{\beta_i^{B,A}(t)}{\beta_i^{B,B}(t)} p_i^A(t) \end{aligned}$$

give rise to the following bounds on prices

$$p_i^A(t) \leq \frac{\alpha_i^A(t)\beta_i^{B,B}(t) + \alpha_i^B(t)\beta_i^{A,B}(t)}{\beta_i^{B,B}(t)\beta_i^{A,A}(t) - \beta_i^{B,A}(t)\beta_i^{A,B}(t)} \quad \text{and}$$

$$p_i^B(t) \leq \frac{\alpha_i^B(t)\beta_i^{A,A}(t) + \alpha_i^A(t)\beta_i^{B,A}(t)}{\beta_i^{B,B}(t)\beta_i^{A,A}(t) - \beta_i^{B,A}(t)\beta_i^{A,B}(t)}.$$

In particular, this shows that the feasible prices are bounded with an upper bound independent of the competitor's strategy. Also, the production rates are bounded by the capacity rate. Moreover, because the time horizon and initial inventory level are finite, and the inventory levels are continuous functions of time, they are bounded as well.

More details on notations and properties of the feasible region are available in the online appendix.

2.2. Objective Function

We will denote by x^k the vector representing a pricing and production strategy along with the state variables of player k : $x^k = (p^k, u^k, I^k)$. The vector (x^A, x^B) represents the collective strategy and state vector.

For a fixed strategy and state vector of the competitor, we denote $Q^k(\bar{x}^{-k})$ the subset of all feasible strategy and state vectors for player k , given the strategy and state vector \bar{x}^{-k} of her competitor. We denote Y the set of collectively feasible strategy and state vectors for both players: $Y = \{x: x^k \in Q^k(x^{-k}), k = A, B\}$. Let J^k be the payoff function of player k . Note it is defined on set X and is real-valued. We recall that for $x = (x^A, x^B) \in X$,

$$J^k(x) = \int_0^T \sum_{i=1}^N (p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k(t)(u_i^k(t))^2 - h_i^k(t)(I_i^k(t))^2) dt, \quad k = A, B.$$

We denote $Q: X \mapsto 2^X$ the mapping such that $Q(x) = Q^A(x^B) \times Q^B(x^A)$ represents the set of feasible collective strategy and state vectors for both players when the competitor keeps her strategy fixed at x^{-k} .

PROPOSITION 1. *Vector x is a Nash equilibrium iff $x \in Y$ and*

$$\forall \bar{x} \in Q(x), \quad J^A(x^A, x^B) + J^B(x^A, x^B) \\ \geq J^A(\bar{x}^A, x^B) + J^B(x^A, \bar{x}^B).$$

The proof of this result and all other results in this paper may be found in the online appendix, unless otherwise specified. The existence of a Nash equilibrium for the deterministic problem using a quasivariational inequalities formulation is shown in the online appendix.

Figure 1. Uncertainty model: Illustration for $\tilde{\alpha}_i^k(t)$.

$$\tilde{\alpha}_i^k(t) = \alpha_i^k(t) + z_i^k(t)\hat{\alpha}_i^k(t), \quad -1 \leq z_i^k(t) \leq 1$$

3. Uncertainty on the Demand: A Robust Optimization Approach

3.1. Model of Uncertainty

In this section, we incorporate demand uncertainty in the model. We consider a general demand model (additive and multiplicative are special cases) and model uncertainty using the robust optimization paradigm (see Adida and Perakis 2006 for an additive demand model with no competition). We consider a model of demand uncertainty with not only coefficients $\alpha_i^k(\cdot)$ uncertain but also the price sensitivities $\beta_i^{k,k}(\cdot)$, $\beta_i^{k,-k}(\cdot)$. We assume that for each supplier k , the realized demand for product i is given by $\tilde{d}_i^k(t) = \tilde{\alpha}_i^k(t) - \tilde{\beta}_i^{k,k}(t)p_i^k(t) + \tilde{\beta}_i^{k,-k}(t)p_i^{-k}(t)$, where the realized values $\tilde{\alpha}_i^k(\cdot)$, $\tilde{\beta}_i^{k,k}(\cdot)$, $\tilde{\beta}_i^{k,-k}(\cdot)$ are uncertain. We do not assume a particular probability distribution, but rather assume that parameters may vary within a range of values that is symmetric around some known nominal value, denoted, respectively, $\alpha_i^k(\cdot)$, $\beta_i^{k,k}(\cdot)$, $\beta_i^{k,-k}(\cdot)$, with respective half-length $\hat{\alpha}_i^k(\cdot)$, $\hat{\beta}_i^{k,k}(\cdot)$, $\hat{\beta}_i^{k,-k}(\cdot)$ (see Figure 1).

Without loss of generality, we assume that $0 < \hat{\alpha}_i^k(\cdot) < \alpha_i^k(\cdot) \forall i, k$, $0 < \hat{\beta}_i^{k,k}(\cdot) < \beta_i^{k,k}(\cdot) \forall i, k$, $0 < \hat{\beta}_i^{k,-k}(\cdot) < \beta_i^{k,-k}(\cdot) \forall i, k$ to ensure that the realized values remain non-negative. The feasible realization must therefore satisfy

$$|\tilde{\alpha}_i^k(t) - \alpha_i^k(t)| \leq \hat{\alpha}_i^k(t), \quad |\tilde{\beta}_i^{k,k}(t) - \beta_i^{k,k}(t)| \leq \hat{\beta}_i^{k,k}(t), \\ |\tilde{\beta}_i^{k,-k}(t) - \beta_i^{k,-k}(t)| \leq \hat{\beta}_i^{k,-k}(t) \quad \forall t, i, k.$$

We do not introduce a particular probability distribution for the uncertain parameters; however, we do not want to take an overly conservative approach that would allow parameters to be at the value corresponding to the worst-case scenario at all times because such a scenario is highly unlikely. We favor a more reasonable and realistic approach that would bound the cumulative deviation from the nominal value. Motivated by this observation, we introduce “budget of uncertainty functions” $\Gamma_i^k(\cdot)$, $\Theta_i^{k,k}(\cdot)$, $\Theta_i^{k,-k}(\cdot)$ taking values on $[0, T]$ that are increasing functions of time, and bound the cumulative dispersion of the realized values around the nominal value over time for, respectively, $\tilde{\alpha}_i^k(\cdot)$, $\tilde{\beta}_i^{k,k}(\cdot)$, $\tilde{\beta}_i^{k,-k}(\cdot)$. The general notion has been used by several authors (see, for example, Bertsimas and Sim 2004 and Bertsimas and Thiele 2006). More details on the motivation for this approach are presented in the online appendix. We assume that $\dot{\Gamma}_i^k(t)$, $\dot{\Theta}_i^{k,k}(t)$, $\dot{\Theta}_i^{k,-k}(t)$

$\geq 0 \forall i, k, t$ because the aggregate dispersion over time can only increase, and that $\Gamma_i^k(t), \Theta_i^{k,k}(t), \Theta_i^{k,-k}(t) \leq 1 \forall i, k, t$ to ensure that the budgets of uncertainty do not grow faster than new variables are added.

The feasible realizations of the parameters must then satisfy the following budget of uncertainty constraints $\forall t, i, k$:

$$\int_0^t \frac{|\tilde{\alpha}_i^k(s) - \alpha_i^k(s)|}{\hat{\alpha}_i^k(s)} ds \leq \Gamma_i^k(t),$$

$$\int_0^t \frac{|\tilde{\beta}_i^{k,k}(s) - \beta_i^{k,k}(s)|}{\hat{\beta}_i^{k,k}(s)} ds \leq \Theta_i^{k,k}(t),$$

$$\int_0^t \frac{|\tilde{\beta}_i^{k,-k}(s) - \beta_i^{k,-k}(s)|}{\hat{\beta}_i^{k,-k}(s)} ds \leq \Theta_i^{k,-k}(t).$$

Denoting

$$z_i^k(t) \equiv \frac{\tilde{\alpha}_i^k(t) - \alpha_i^k(t)}{\hat{\alpha}_i^k(t)}, \quad y_i^{k,k}(t) \equiv \frac{\tilde{\beta}_i^{k,k}(t) - \beta_i^{k,k}(t)}{\hat{\beta}_i^{k,k}(t)},$$

$$y_i^{k,-k}(t) \equiv \frac{\tilde{\beta}_i^{k,-k}(t) - \beta_i^{k,-k}(t)}{\hat{\beta}_i^{k,-k}(t)}$$

the scaled variations, it follows that the constraints can be rewritten as $z_i^k(t), y_i^{k,k}(t), y_i^{k,-k}(t) \in [-1, 1]$ and

$$\int_0^t |z_i^k(s)| ds \leq \Gamma_i^k(t), \quad \int_0^t |y_i^{k,k}(s)| ds \leq \Theta_i^{k,k}(t),$$

$$\int_0^t |y_i^{k,-k}(s)| ds \leq \Theta_i^{k,-k}(t) \quad \forall t, i, k.$$

Uncertainty set \mathcal{F}^k contains all realizations $(\tilde{\alpha}^k, \tilde{\beta}^{k,k}, \tilde{\beta}^{k,-k})$ satisfying the constraints above. It follows that for feasible realizations, $\int_0^t |z_i^k(s)| ds, \int_0^t |y_i^{k,k}(s)| ds, \int_0^t |y_i^{k,-k}(s)| ds \leq t$.

REMARK. This implies in particular, that if for some time t , $\Gamma_i^k(t) \geq t$, the budget of uncertainty constraint $\int_0^t |z_i^k(s)| ds \leq \Gamma_i^k(t)$ follows from the bounds imposed on $z_i^k(\cdot)$. As a result, at the optimal solution to the robust optimization problem, $|z_i^k(\cdot)|$ will be equal to one on $[0, t]$ because this corresponds to the worst-case scenario on that interval and it is allowed by the budget of uncertainty constraint. In particular, the exact value of $\Gamma_i^k(t)$ (given that it is greater than or equal to t) will not matter in that case. We conclude from this remark that the *effective* budget of uncertainty for $\alpha_i^k(\cdot)$ is $\min\{t, \Gamma_i^k(t)\}$ as was introduced in Adida and Perakis (2006). To measure the global uncertainty of the problem, we want to introduce a quantity used as a metric representative of the budgets of uncertainty. As a result, instead of simply using the integral of $\Gamma_i^k(\cdot)$ over the time horizon, we introduce the cumulative effective budget of uncertainty defined by $\int_0^T \min\{t, \Gamma_i^k(t)\} dt$, and similarly for the other parameters.

The goal of the robust formulation is then to find the solution that maximizes the *nominal* objective value and that is feasible for any feasible realization of the parameters. Nevertheless, as we discuss in Adida and Perakis (2006), we could also have analyzed using similar ideas as below, the case of maximizing the “worst-case” objective value for any feasible realization of the parameters.

3.2. Robust Formulation, Properties, and Existence Result

Taking into account uncertainty, the robust formulation of the best-response problem faced by supplier k can be written as follows (we omit the time argument for the sake of clarity in each of the terms below that are all time dependent—except the initial inventory level):

$$\max_{p^k(\cdot), u^k(\cdot)} \int_0^T \sum_{i=1}^N (p_i^k(\alpha_i^k - \beta_i^{k,k} p_i^k + \beta_i^{k,-k} p_i^{-k}) - \gamma_i^k (u_i^k)^2 - h_i^k (I_i^k)^2) dt \quad (5)$$

s.t. $\dot{I}_i^k = u_i^k - \alpha_i^k + \beta_i^{k,k} p_i^k - \beta_i^{k,-k} p_i^{-k}$
 $\forall t \in [0, T] \forall i, I_i^k(0) = I_i^0, \quad (6)$

$$\sum_{i=1}^N u_i^k \leq K^k \quad \forall t \in [0, T],$$

$$u_i^k, p_i^k \geq 0 \quad \forall t \in [0, T] \forall i,$$

$$\forall (\tilde{\alpha}^k, \tilde{\beta}^{k,k}, \tilde{\beta}^{k,-k}) \in \mathcal{F}^k,$$

$$\dot{I}_i^k = u_i^k - \tilde{\alpha}_i^k + \tilde{\beta}_i^{k,k} p_i^k - \tilde{\beta}_i^{k,-k} p_i^{-k} \quad \forall t \in [0, T] \forall i, \tilde{I}_i^k(0) = I_i^0, \quad (7)$$

$$p_i^k \leq \frac{\tilde{\alpha}_i^k + \tilde{\beta}_i^{k,-k} p_i^{-k}}{\tilde{\beta}_i^{k,k}} \quad \forall t \in [0, T] \forall i, \quad (8)$$

$$\tilde{I}_i^k \geq 0 \quad \forall t \in [0, T], \forall i. \quad (9)$$

The fluid Equations (6) and (7) describe, respectively, the evolution of the nominal and realized inventory levels. Equation (9) reflects the no backorder constraint on the realized inventory level. The nonnegativity of the realized demand leads to an upper bound on prices (Equation (8)). Finally, the objective function involves the nominal inventory level. Note that this problem includes an infinite number of constraints not only because constraints have to be satisfied continuously with time, but also because the constraint involving uncertain parameters must be satisfied for any feasible realization.

THEOREM 1. *Reintroducing the time argument, the robust counterpart of problem (5) is*

$$\max \int_0^T \sum_{i=1}^N (p_i^k(t)(\alpha_i^k(t) - \beta_i^{k,k}(t)p_i^k(t) + \beta_i^{k,-k}(t)p_i^{-k}(t)) - \gamma_i^k (u_i^k(t))^2 - h_i^k(t)(I_i^k(t))^2) dt$$

s.t. $\dot{I}_i^k(t) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_i^k(t) - \beta_i^{k,-k}(t)p_i^{-k}(t)$
 $\forall i \forall t \in [0, T], I_i^k(0) = I_i^0,$

$$\sum_{i=1}^N u_i^k(t) \leq K^k(t) \quad \forall t \in [0, T],$$

$$p_i^k(t) \leq \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))p_i^{-k}(t)}{\hat{\beta}_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)}$$

$$\forall i \forall t \in [0, T],$$

$$I_i^k(t) \geq \Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot)) \quad \forall i \forall t \in [0, T],$$

$$p_i^k(t), u_i^k(t) \geq 0 \quad \forall i \forall t \in [0, T],$$

where $\Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot))$ is given by

$$\begin{aligned} \Omega_i^k(t) = & \min_{\omega_i^k(t), r_i^k(s, t)} \omega_i^k(t) \Gamma_i^k(t) + \int_0^t r_i^k(s, t) ds \\ \text{s.t. } & \omega_i^k(t) + r_i^k(s, t) \geq \hat{\alpha}_i^k(s), \\ & \omega_i^k(t) \geq 0, r_i^k(s, t) \geq 0 \quad \forall s \in [0, t] \end{aligned} \quad (10)$$

and $\Omega_i^{k^2}(t, p^k(\cdot)), \Omega_i^{k^3}(t, p^{-k}(\cdot))$ are defined similarly after substituting, respectively, $\hat{\beta}_i^{k,k}(\cdot)p_i^k(\cdot), \hat{\beta}_i^{k,-k}(\cdot)p_i^{-k}(\cdot)$ for $\hat{\alpha}_i^k(\cdot)$, and $\Theta_i^{k,k}(\cdot), \Theta_i^{k,-k}(\cdot)$ for $\Gamma_i^k(\cdot)$.

Note that this problem is deterministic: it does not involve any uncertain parameter realization. It still has an infinite number of constraints, but only because of time continuity. The problem written in this form is intuitive in the sense that the uncertainty on the demand parameters translates into a tighter upper bound on the prices and on a minimum inventory level guarantee. As a result, even in the presence of data perturbation (as defined by the budget of uncertainty and ranges of variation), these stronger constraints guarantee that the bounds will be satisfied.

The uncertainty of demand thus translates into protection levels for the prices (via an upper bound) and for the inventory levels (via a lower bound) that are stronger than in the nominal case so that, even with some variation in the demand parameters—within the introduced uncertainty constraints—the realized inventory levels will remain positive, and realized demand will remain nonnegative.

In the online appendix, we show how each of the three dual subproblems can be solved. Even though there is no closed-form solution in general, we show how to calculate the objective value of the dual subproblems as a function of the data and the prices.

Referring to the online appendix, the reader may notice that the notion of effective budget of uncertainty we introduced is validated. Indeed, the robust counterpart involves the actual value of the budget of uncertainty for each parameter only if the budget of uncertainty at time t is smaller than t . Otherwise, the robust formulation does not depend on it, and the security level requirement corresponds to the worst-case scenario on $[0, t]$. In the case where the actual budget of uncertainty does matter (i.e., has a value smaller than t), then the security level is lower, thus easier to satisfy, which makes the solution less conservative and yields a higher objective.

In particular, we note from the online appendix that

$$\begin{aligned} \Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot)) \\ \leq \int_0^t (\hat{\alpha}_i^k(s) + \hat{\beta}_i^{k,k}(s)p_i^k(s) + \hat{\beta}_i^{k,-k}(s)p_i^{-k}(s)) ds. \end{aligned} \quad (11)$$

PROPOSITION 2. *The inventory security level $\Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot))$ is nondecreasing with time.*

PROOF. We prove this result by showing that $\Omega_i^{k^1}, \Omega_i^{k^2}$, and $\Omega_i^{k^3}$ are each nondecreasing with time. The proof is given in detail in the online appendix for $\Omega_i^{k^1}$, and it is similar for $\Omega_i^{k^2}$ and $\Omega_i^{k^3}$. \square

PROPOSITION 3. *The minimum inventory security level $\Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot))$ is nondecreasing with each budget of uncertainty $\Gamma_i^k(t), \Theta_i^{k,k}(t), \Theta_i^{k,-k}(t)$.*

These properties make sense at an intuitive level because the more uncertainty in the problem, the higher the protection level should be. Moreover, we observe that the minimum inventory level is also nondecreasing in the half-lengths of allowed ranges of variation $\hat{\alpha}_i^k(\cdot), \hat{\beta}_i^{k,k}(\cdot), \hat{\beta}_i^{k,-k}(\cdot)$, which makes sense for the same reason. Furthermore, the cumulative uncertainty over time can only increase, therefore the protection level should also increase over time.

PROPOSITION 4. *The minimum inventory security level $\Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot))$ is convex in $p_i^k(\cdot)$ (and thus, by symmetry, in $p_i^{-k}(\cdot)$).*

COROLLARY 1. *For a fixed $p^{-k}(\cdot)$, the feasible set of the robust counterpart above is convex.*

PROOF. All constraints are linear except the minimum inventory security level constraint. This constraint can be written as $\Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot)) - I_i^k(t) \leq 0$. Because $\Omega_i^k(t, p_i^k(\cdot), p_i^{-k}(\cdot))$ is convex in terms of the prices, the left-hand side is convex in the variables, which yields the result. \square

In this formulation of the robust counterpart, we have a deterministic fluid model with no more variables than the initial problem. Nevertheless, the constraints are no longer linear. However, we still have a convex optimization problem because we have a concave quadratic objective maximized over a convex set. We can show the following using the upper bound on feasible prices described in the previous section.

PROPOSITION 5. *Prices $p_i^k(t)$ are bounded from above by*

$$\begin{aligned} p_{i_{\max}}^k(t) = & ((\alpha_i^k(t) - \hat{\alpha}_i^k(t))(\beta_i^{-k,-k}(t) + \hat{\beta}_i^{-k,-k}(t)) \\ & + (\alpha_i^{-k}(t) - \hat{\alpha}_i^{-k}(t))(\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))) \\ & \cdot ((\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t))(\beta_i^{-k,-k}(t) + \hat{\beta}_i^{-k,-k}(t)) \\ & - (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t))(\beta_i^{-k,k}(t) - \hat{\beta}_i^{-k,k}(t)))^{-1}. \end{aligned}$$

In particular, all control and state variables are bounded. (The production rates are bounded above by the production capacity rate, and the inventory levels evolve continuously on a finite time horizon from a finite initial value.)

ASSUMPTION 2. We assume that

$$2 \sum_{i=1}^N (\hat{\alpha}_i^k(t) + \hat{\beta}_i^{k,k}(t) p_{i_{\max}}^k(t) + \hat{\beta}_i^{k,-k}(t) p_{i_{\max}}^{-k}(t)) \leq K^k(t) \quad \forall t, k = A, B.$$

This assumption guarantees that the feasible set is not empty (see the online appendix). More specifically, it ensures that the capacity level is sufficient to guarantee the feasibility of the problem. Intuitively, the larger the range of allowed variation for the parameters, the higher the security level for the inventory, and the lower the upper bound on prices. To stop the inventory from decreasing and going below its minimum security level, we can either increase production or increase prices. Because the prices are bounded from above, the more uncertainty, the higher the production rates need to be to satisfy such a guarantee. This is the reason why we have to impose a minimum production capacity available. In other words, if the production capacity level is too low, we are not able to immune the solution against too much uncertainty on data parameters.

THEOREM 2. Under Assumptions 1 and 2, there exists a Nash equilibrium to the robust formulation.

We now wish to provide and analyze a practical method for computing an equilibrium. As a result, we discretize time. We first discuss the uniqueness of Nash equilibria in this setting and subsequently, we devise an algorithm for determining a Nash equilibrium.

4. Discretized Robust Formulation

We rewrite the best-response problem for player k by discretizing time. To improve the exposition, we consider without loss of generality a time step length of one and assume that T is an integer, such that the discrete time instants are $t = 0, 1, \dots, T$. The first decisions are made at time $t = 1$. For each player, we now have $2NT$ control variables (prices and production rates for all products at all times) and NT state variables (inventory levels).

We proved in §3 that the robust counterpart is equivalent to formulation (12):

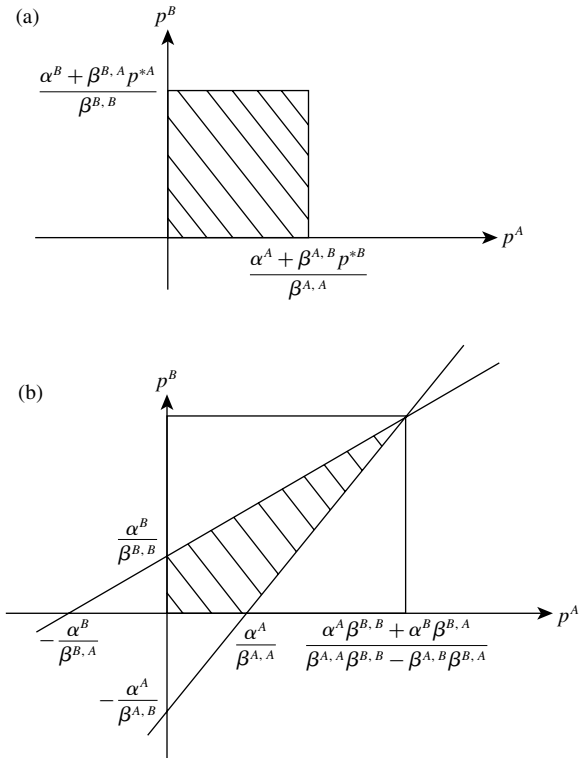
$$\max_{u_i^k(t), p_i^k(t), \forall i, t} \sum_{t=1}^T \sum_{i=1}^N (p_i^k(t) (\alpha_i^k(t) - \beta_i^{k,k}(t) p_i^k(t) + \beta_i^{k,-k}(t) p_i^{-k}(t)) - \gamma_i^k(t) (u_i^k(t))^2 - h_i^k(t) (I_i^k(t))^2) \quad (12)$$

$$\text{s.t. } I_i^k(t) - I_i^k(t-1) = u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t) p_i^k(t) - \beta_i^{k,-k}(t) p_i^{-k}(t), \quad t = 1, \dots, T, i = 1, \dots, N,$$

$$I_i^k(0) = I_i^k, \quad i = 1, \dots, N,$$

$$\sum_{i=1}^N u_i^k(t) \leq K^k(t), \quad t = 1, \dots, T,$$

Figure 2. Example 1 in the space of prices: (a) set $Q(p^{*A}, p^{*B})$ and (b) set Y .



$$p_i^k(t) \leq \frac{\alpha_i^k(t) - \hat{\alpha}_i^k(t) + (\beta_i^{k,-k}(t) - \hat{\beta}_i^{k,-k}(t)) p_i^{-k}(t)}{\beta_i^{k,k}(t) + \hat{\beta}_i^{k,k}(t)}, \quad t = 1, \dots, T, i = 1, \dots, N,$$

$$I_i^k(t) \geq \Omega_i^k(t, p^k(\cdot), p^{-k}(\cdot)), \quad t = 1, \dots, T, i = 1, \dots, N,$$

$$u_i^k(t), p_i^k(t) \geq 0, \quad t = 1, \dots, T, i = 1, \dots, N.$$

The notations and properties of the problem that were shown above in a continuous-time setting clearly extend to the discretized setting. Vector x^k of control and state variables for supplier k has $3NT$ components $x^k = (p_i^k(t), u_i^k(t), I_i^k(t), t = 1, \dots, T, i = 1, \dots, N)$ and $x = (x^A, x^B) \in \mathbb{R}^{6NT}$. We will use the usual norm on this vector space. Because the dimension is now finite, it follows that Y is a compact set.

EXAMPLE 1. We illustrate the sets Y and $Q(x)$ on a simple example. Let us simplify our problem and consider the case of one product and one time period in the space of prices only. The best-response problem for supplier k is then written as follows:

$$\max_{p^k} p^k (\alpha^k - \beta^{k,k} p^k + \beta^{k,-k} p^{-k})$$

$$\text{s.t. } 0 \leq p^k \leq \frac{\alpha^k + \beta^{k,-k} p^{-k}}{\beta^{k,k}}.$$

Figure 2 illustrates the sets $Q(p^{*A}, p^{*B}) = \{(p^A, p^B) \geq 0: p^A \leq (\alpha^A + \beta^{A,B} p^{*B})/\beta^{A,A}, p^B \leq (\alpha^B + \beta^{B,A} p^{*A})/\beta^{B,B}\}$, $Y = \{(p^A, p^B) \geq 0: \alpha^A - \beta^{A,A} p^A + \beta^{A,B} p^B \geq 0, \alpha^B - \beta^{B,B} p^B + \beta^{B,A} p^A \geq 0\}$. The following existence result was proven by Rosen (1965) based on Kakutani's fixed-point theorem, using the facts that the joint strategy set is closed, convex, and bounded, and the payoff function of a player is continuous in all arguments and concave with the player's strategy vector.

THEOREM 3 (ROSEN 1965). *An equilibrium point exists for every concave n -person game.*

We introduce the Nikaido-Isoda function $\psi(x, z) = \sum_{k=A, B} (J^k(z^k, x^{-k}) - J^k(x^k, z^{-k}))$, $x, z \in \mathbb{R}^{6NT}$. Notice that if $x \in Y, z \in Q(x)$, this expression represents the sum of the changes in the players' payoff function when they change unilaterally their strategy and state from x^k to z^k while the other player keeps strategy and state x^{-k} . In the expression below we omit the time argument to simplify the reading:

$$\begin{aligned} \psi(x, \bar{x}) &= \sum_{t=1}^T \sum_{i=1}^N \sum_{k=A, B} [\bar{p}_i^k (\alpha_i^k - \beta_i^{k,k} \bar{p}_i^k + \beta_i^{k,-k} p_i^{-k}) - \gamma_i^k \bar{u}_i^{k^2} - h_i^k \bar{I}_i^{k^2} \\ &\quad - p_i^k (\alpha_i^k - \beta_i^{k,k} p_i^k + \beta_i^{k,-k} p_i^{-k}) + \gamma_i^k u_i^{k^2} + h_i^k I_i^{k^2}] \\ &= \sum_{t=1}^T \sum_{i=1}^N \sum_{k=A, B} [(\alpha_i^k + \beta_i^{k,-k} p_i^{-k})(\bar{p}_i^k - p_i^k) - \beta_i^{k,k} (\bar{p}_i^{k^2} - p_i^{k^2}) \\ &\quad - \gamma_i^k (\bar{u}_i^{k^2} - u_i^{k^2}) - h_i^k (\bar{I}_i^{k^2} - I_i^{k^2})]. \end{aligned}$$

We observe that ψ is a continuous function in each one of its arguments. It is clear from the definition that $\psi(x, x) = 0 \forall x \in \mathbb{R}^{6NT}$. Consider a fixed $x \in Y$. Because $x \in Q(x)$, we have $\max_{z \in Y} \psi(x, z) \geq 0, \max_{z \in Q(x)} \psi(x, z) \geq 0$.

PROPOSITION 6. *$x^* \in Y$ is a Nash equilibrium if and only if $\max_{z \in Q(x^*)} \psi(x^*, z) = 0$.*

In other words, we showed that x^* is a Nash equilibrium if and only if $x^* = \text{Arg max}_{z \in Q(x^*)} J^A(z^A, x^{B*}) + J^B(x^{A*}, z^B)$. This property presents the advantage of combining into a single condition the usual definition of a Nash equilibrium that involves conditions for each player. Note that the set we are maximizing over depends on x^* . We now introduce normalized Nash equilibria as a particular case of the ones defined by Rosen (1965).

DEFINITION 1. We will refer to x^* as a *normalized Nash equilibrium* iff $\max_{z \in Y} \psi(x^*, z) = 0$.

In other words, in this paper, x^* is a normalized Nash equilibrium if and only if $x^* = \text{Arg max}_{z \in Y} J^A(z^A, x^{B*}) + J^B(x^{A*}, z^B)$. Note that in this definition the maximum is taken over Y and not $Q(x^*)$ like for a Nash equilibrium. This means that we consider responses that are *jointly feasible*, and not responses that are unilaterally feasible given the current competitor's strategy.

LEMMA 1 (ROSEN 1965). *A normalized Nash equilibrium is a Nash equilibrium.*

As a side remark, Rosen (1965) showed that the converse is true when the feasible strategy set is not a coupled constraint set, i.e., when the feasible strategy of each player is independent of the competitors' strategies. Then, for a coupled constraint game, a Nash equilibrium is not necessarily a normalized Nash equilibrium (when there are multiple Nash equilibria).

The normalized Nash equilibrium can be interpreted qualitatively as the Nash equilibrium where the coupling constraints have the same shadow price (or Lagrange multiplier) in the best-response problem faced by each player. In this sense, the normalized Nash equilibrium may seem "fair" because both players contribute equally to the global constraints, relative to the marginal profit loss. As a result, this Nash equilibrium may be the most desirable to attain. Moreover, in some settings, coupling constraints may be due to social requirements that a central authority wants to impose, but players have no direct incentive to achieve. It is therefore of interest to design a way to give incentives to the players in order for them to reach the normalized Nash equilibrium, assuming that they will ignore the coupling constraints, and thus they participate in a decoupled game.

To this end, Haurie (1995) presents a tax scheme that modifies the payoff functions and leads the decoupled game to have as a Nash equilibrium the original normalized Nash equilibrium, and achieves the coupling constraints. In this paper, the normalized Nash equilibrium is calculated by assigning the same weight to the two players. Therefore, the authority that imposes the taxes treats the two players in the same way and as equally responsible of the burden of achieving the coupling constraints. The tax is designed such that, assuming the agents solve a best-response problem ignoring coupled constraints, and considering only noncoupled constraints, the Nash equilibrium corresponds to the normalized Nash equilibrium under the presence of the coupled constraints. In this paper, the coupled constraints for a player are the nonnegativity of her inventory levels and of her demand rates (that imply an upper bound on her prices). The normalized Nash equilibrium is then the Nash equilibrium that shares equally the burden of satisfying each one of those coupled constraints, i.e., keeping inventory levels and demand rates of both players, all products, at all times, nonnegative.

Notice that in Example 1, we can show that there exists a unique Nash equilibrium $p^* = (p^{*A}, p^{*B})$ that is also the unique normalized Nash equilibrium and is located in the interior of Y and in the center of the rectangle representing $Q(p^*)$. This equilibrium is given by solving $p^{*A} = (\alpha^A + \beta^{A,B} p^{*B})/(2\beta^{A,A}), p^{*B} = (\alpha^B + \beta^{B,A} p^{*A})/(2\beta^{B,B})$, which leads to $p^{*A} = (2\beta^{B,B} \alpha^A + \beta^{A,B} \alpha^B)/(4\beta^{A,A} \beta^{B,B} - \beta^{A,B} \beta^{B,A}), p^{*B} = (2\beta^{A,A} \alpha^B + \beta^{B,A} \alpha^A)/(4\beta^{A,A} \beta^{B,B} - \beta^{A,B} \beta^{B,A})$.

EXAMPLE 2. We are providing a simple example inspired from Rosen (1965) to show the difference between a Nash equilibrium and a normalized Nash equilibrium in the case of a coupled constraint strategy set. Consider two players A and B who must decide their respective strategies $x, y \in \mathbb{R}$ subject to the noncoupling constraints $x \geq 0, y \geq 0$ and the coupling constraint $x + y \geq 1$, with payoff functions $J^A(x, y) = -\frac{1}{2}x^2 + xy, J^B(x, y) = -y^2 - xy$. In this example, $Y = \{x, y \geq 0: x + y \geq 1\}$, while $Q(x^0, y^0) = [m(y^0), \infty) \times [m(x^0), \infty)$ where $m(z) = \max\{0, 1 - z\}$. We first observe that $(x, 1 - x)$ is a Nash equilibrium for all $x \in [\frac{1}{2}, 1]$ because $\text{Arg max}_{x: x \geq 0, x \geq 1 - y^0} J^A(x, y^0) = 1 - y^0$ if $y^0 \leq \frac{1}{2}$, $\text{Arg max}_{y: y \geq 0, y \geq 1 - x^0} J^B(x^0, y) = 1 - x^0$. Let us now find a normalized Nash equilibrium by solving $\max J^A(x, y^0) + J^B(x^0, y) = -\frac{1}{2}x^2 + xy^0 - y^2 - x^0y$ subject to $x + y \geq 1, x, y \geq 0$. It is easy to verify that the solution is $x = y = 0$ if $y^0 > 1$, and $x = 1, y = 0$ otherwise. In particular, $(x^0 = 1, y^0 = 0)$ is the only normalized Nash equilibrium.

Therefore in this example, there are infinitely many Nash equilibria, but there is a unique normalized Nash equilibrium that is included in the set of Nash equilibria.

5. Existence and Uniqueness of a Normalized Nash Equilibrium

THEOREM 4 (ROSEN 1965). *There exists a normalized Nash equilibrium point to a concave n -person game. In particular, there exists a normalized Nash equilibrium point for the problem we are considering in this paper.*

We observe that set Y can be written as a set of inequalities $\{x: H(x) \geq 0\}$ with H being concave. We argue that the following constraint qualification holds: $\exists x \in Y: H_j(x) > 0 \forall j$. This can be shown in a way similar to the proof of Y nonempty. We also notice that $H_j(x)$ possesses continuous first derivatives for $x \in Y$ in formulation (11) (available in the online appendix) because all constraints are linear.

For a scalar function $J(x)$, we will denote by $\nabla_{x^k} J(x)$ the gradient of $J(x)$ with respect to x^k . Thus, $\nabla_{x^k} J(x) \in \mathbb{R}^{3NT}$. We denote $\sigma(x) = J^A(x) + J^B(x)$ and $g(x) = (\nabla_{x^A} J^A(x), \nabla_{x^B} J^B(x))$. $g(x)$ is called the pseudogradient of $\sigma(x)$.

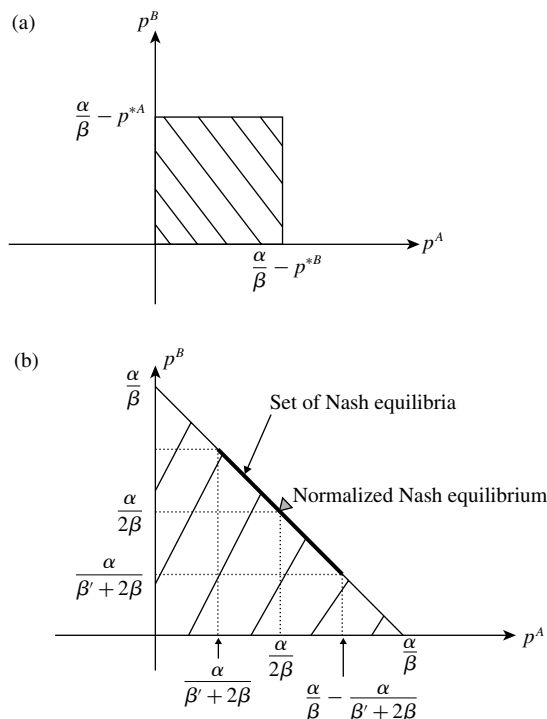
DEFINITION 2. The function $\sigma(x) = J^A(x) + J^B(x)$ is called diagonally strictly concave for $x \in Y$ if for every $x^0, x^1 \in Y$ we have $(x^1 - x^0)'g(x^0) + (x^0 - x^1)'g(x^1) > 0$.

Using the Karush-Kuhn-Tucker conditions, Rosen shows the following:

THEOREM 5 (ROSEN 1965). *If $\sigma(x)$ is diagonally strictly concave, then the normalized Nash equilibrium point is unique.*

Note that the strict diagonal concavity of $\sigma(x)$, which here implies uniqueness of the normalized Nash equilibrium for coupled constraint games, would imply uniqueness

Figure 3. Modified example in space of prices: (a) set $Q(p^{*A}, p^{*B})$ and (b) set Y .



of the Nash equilibrium if the constraints were not coupled. For coupled constraint games, the Nash equilibrium is in general not unique. We illustrate this in the following example.

EXAMPLE 1 MODIFIED. We now modify Example 1 to show that there may be multiple Nash equilibria but a unique normalized Nash equilibrium when a Nash equilibrium lies on the boundary of the feasible set. We assume here that all data are symmetric between A and B , and simplify the notation as follows: $\beta^{A,A} = \beta^{B,B} = \beta, \beta^{A,B} = \beta^{B,A} = \beta', \alpha^A = \alpha^B = \alpha$. We add the constraint $p^A + p^B \leq \alpha/\beta$, which is not satisfied by the Nash equilibrium in Example 1. The set Y and $Q(p^{*A}, p^{*B})$ for the modified example are illustrated in Figure 3. We can show that there is an infinite number of Nash equilibria as indicated in Figure 3 that correspond to prices such that $p^A + p^B = \alpha/\beta$ and $\alpha/(\beta' + 2\beta) \leq p^k \leq \alpha/\beta - \alpha/(\beta' + 2\beta), k = A, B$. The unique normalized Nash equilibrium is $p^{*A} = p^{*B} = \alpha/(2\beta)$.

Intuitively, what makes the modified problem have a normalized Nash equilibrium that is not the unique Nash equilibrium is the fact that a coupling constraint is binding at the normalized equilibrium. Indeed, in that case, the optimization problem determining the normalized Nash equilibrium is not separable into the two subproblems that determine the Nash equilibrium. Notice in particular that in Example 1 the unique Nash equilibrium is in the interior of Y , while in Example 2 and Example 1 modified

there are multiple Nash equilibria that lie on the boundary of Y . In all three examples, however, there was, as expected, a unique normalized Nash equilibrium.

THEOREM 6. *Under Assumptions 1 and 2, the normalized Nash equilibrium point is unique.*

To prove this theorem, we use a result from Rosen (1965). Let $G(x)$ be the Jacobian of $g(x)$.

THEOREM 7 (ROSEN 1965). *A sufficient condition that $\sigma(x)$ be diagonally strictly concave for $x \in Y$ is that the symmetric matrix $G(x) + G^T(x)$ be negative definite for $x \in Y$.*

The proof of Theorem 6 can be found in the online appendix. To summarize, although the Nash equilibrium is in general not unique, the normalized Nash equilibrium is unique. Hence, we will focus on how to compute the normalized Nash equilibrium via the algorithm presented next.

6. Solution Algorithm

We consider an iterative relaxation algorithm in which at each iteration, a current pricing policy is given for both suppliers, and the suppliers respond by determining simultaneously a new strategy such that (1) the objective function involves the *current* given pricing policy of her competitor, and (2) in the feasibility constraints, the responses must be jointly feasible. The intuition is to reflect actual practices in which suppliers adapt their policy to improve their performance based on their competitor's current strategy, but because they do so simultaneously they are constrained by the competitor's reaction. As we will show, this algorithm converges to the unique normalized Nash equilibrium, which as we proved above is a particular Nash equilibrium.

We define the simultaneous best-response function $\text{BR}(x): Y \mapsto Y$ by $\text{BR}(x) = \text{Arg max}_{z \in Y} \psi(x, z)$ (or equivalently, $\text{BR}(x) = \text{Arg max}_{z \in Y} J^A(z^A, x^B) + J^B(x^A, z^B)$).

Therefore, the unique normalized Nash equilibrium is defined as the fixed point of map BR . $\text{BR}(x)$ represents the vector with components the respective optimal strategies each player should simultaneously adopt (and are thus jointly feasible because the maximization is taken over set Y) when their payoffs are based on the fact that the other player keeps strategy x^{-k} . We observe that to compute the optimum response, we only need to solve a constrained concave maximization quadratic program with a convex bounded nonempty set, with $6NT$ variables, $6NT$ equality constraints, and $8NT + 2T$ inequality constraints. Therefore, the solution $\text{BR}(x)$ exists, is uniquely defined, and can be determined very fast using standard optimization software.

We introduce a relaxation algorithm to find a normalized Nash equilibrium:

Algorithm \mathcal{A}

Start at $x_0 \in Y$; let $s = 0$. Iteration: Let $x_{s+1} = (1 - \delta_s)x_s + \delta_s \text{BR}(x_s)$ and do $s \leftarrow s + 1$. If $\|x_s - \text{BR}(x_s)\| < \eta$, stop. Else, do a new iteration.

We will use this as a stopping criterion with $\eta = 10^{-6}$. We choose step sizes $\delta_s \in (0, 1)$, $s = 0, 1, \dots$, such that

$$\begin{aligned} \delta_s > 0, \quad s = 0, 1, \dots, \\ \sum_{s=0}^{\infty} \delta_s = \infty, \quad \delta_s \rightarrow 0 \text{ as } s \rightarrow \infty. \end{aligned} \quad (13)$$

We observe that because the starting point x_0 is chosen in set Y , and because $\text{BR}(x) \in Y$ for $x \in Y$, the sequence of vectors produced by this algorithm is in set Y . We showed that the vector such that $\forall i, t, p_i^k(t) = p_{i_{\max}}^k(t)$, $u_i^k(t) = 2\hat{\alpha}_i^k(t) + 2\hat{\beta}_i^{k,k}(t)p_{i_{\max}}^k(t) + 2\hat{\beta}_i^{k,-k}(t)p_{i_{\max}}^{-k}(t)$, $I_i^k(t) = I_i^{t-1} + u_i^k(t) - \alpha_i^k(t) + \beta_i^{k,k}(t)p_{i_{\max}}^k(t) - \beta_i^{k,-k}(t)p_{i_{\max}}^{-k}(t)$ are feasible vectors, and we will use it as a possible starting point for Algorithm \mathcal{A} .

THEOREM 8 (URYAS'EV AND RUBINSTEIN 1994). *If*

1. Y is a convex compact subset of \mathbb{R}^{6NT} ;
2. the function ψ is a continuous, weakly, convex concave function and $\psi(x, x) = 0$, $x \in Y$;
3. the optimum response function $\text{BR}(\cdot)$ is single valued and continuous;
4. the residual terms v and μ (defined below) satisfy the inequality $v(x, y) - \mu(y, x) \geq \zeta(\|x - y\|)$, $x, y \in Y$, where ζ is a strictly monotonically increasing function such that $\zeta(0) = 0$; and

5. the step sizes satisfy condition (13);
then Algorithm \mathcal{A} described above converges to the unique normalized Nash equilibrium.

To apply this theorem and prove convergence of Algorithm \mathcal{A} , we first show some properties. The following result is obtained in a way similar to Daniel (1973).

PROPOSITION 7. *BR is continuous on Y .*

DEFINITION 3. A function $f(\cdot): S \mapsto \mathbb{R}$ is said to be weakly convex on a convex set S if and only if $\forall x, y \in S, 0 \leq \theta \leq 1$, we have $\theta f(x) + (1 - \theta)f(y) \geq f(\theta x + (1 - \theta)y) + \theta(1 - \theta)v(x, y)$, where the remainder $v: S \times S \mapsto \mathbb{R}$ satisfies $(v(x, y))/\|x - y\| \rightarrow 0$ as $x \rightarrow w, y \rightarrow w$ for all $w \in S$.

A function $f(\cdot): S \mapsto \mathbb{R}$ is said to be weakly concave on a convex set S if and only if the function $-f(\cdot)$ is weakly convex on S .

PROPOSITION 8. *The Nikaido-Isoda function $\psi(x, y)$ is weakly convex on Y in its first argument.*

PROPOSITION 9. *The Nikaido-Isoda function $\psi(x, y)$ is concave on Y in its second argument.*

ψ is thus said to be weakly convex concave.

PROPOSITION 10. *There exists a strictly monotonically increasing function $\zeta: \mathbb{R} \mapsto \mathbb{R}$ such that $\zeta(0) = 0$ and $v(x, y) - \mu(y, x) \geq \zeta(\|x - y\|)$, $x, y \in Y$.*

We have shown that all the conditions in Theorem 8 hold for our problem.

COROLLARY 2. *Under Assumptions 1 and 2, Algorithm \mathcal{A} converges to the unique normalized Nash equilibrium (which is a particular Nash equilibrium).*

7. Numerical Results

We implement the robust competitive reformulation and Algorithm \mathcal{A} on a time horizon $T = 10$. Our goal is to study the role of input parameters on the equilibrium solution. To isolate the effect of these parameters, we will implement the algorithm for one product $N = 1$. We will thus omit the subscript i in this section. We chose δ_s as the constant 0.99 for the first 50 iterations, and then equal to $1/(s - 49)$. (The algorithm stopped in most cases in less than 20 iterations.)

7.1. Deterministic Formulation

We first consider an example with no data uncertainty. We want to study the effect of the price sensitivities (coefficients $\beta(\cdot)$). The effect of the capacity level $K(\cdot)$ and the initial inventory level I^0 are shown in the online appendix.

We will consider on the one hand price sensitivities that are symmetric for the suppliers (scenarios a–d), i.e., $\beta^{A,A}(\cdot) = \beta^{B,B}(\cdot)$ and $\beta^{A,B}(\cdot) = \beta^{B,A}(\cdot)$, and on the other hand asymmetric price sensitivities (scenarios e–j). In both cases we will consider successively:

1. Demand sensitivities to prices that increase with time (i.e., when closer to the end time, an increase in prices affects the demand more): scenarios a, b, e–g.

2. Demand sensitivities to prices that decrease with time (i.e., products whose high price matters less at the end of the horizon): scenarios c, d, h–j.

Price sensitivities that increase with time correspond to products that become less attractive to the customer toward the end of the time horizon, for example, products subject to a seasonality effect, or such that there have appeared on the market newer products that can serve as a substitute. Price sensitivities that decrease with time correspond to products that become more attractive to the customer toward the end of the time horizon, for example, because of a marketing campaign or an appearing trend. Also in both cases, we will choose the data such that the ratio $\rho^k = \beta^{k,-k}/\beta^{k,k}$ is constant across time. In the symmetric case, this ratio will clearly be the same for both suppliers. In the asymmetric case, we will consider inputs such that either this ratio is the same for the two suppliers (with two values), or it is different. We use the following parameter values for $k = A, B$, $t = 1, \dots, 10$: $\alpha^k(t) = 15$, $\gamma^k(t) = 0.01$, $h^k(t) = 0.01$, $I^0 = 10$, capacity $K^k(t) = 10$ and demand sensitivities to prices as shown in the online appendix (we verified that Assumption 1 holds in all cases). One must be careful when interpreting the objective values across scenarios because the demand sensitivities have an effect on the total demand. The results are shown in Figure 4(a) and in the online appendix.

The insights are summarized as follows. Prices evolve with time with a trend opposite to the price sensitivities, i.e., prices are higher when the sensitivities are lower. In most cases (when the sensitivities are not too high), the inventory levels decrease from the initial value to zero, and

then remain at that level. Production rates are adjusted to maintain the zero inventory level. When the price sensitivities are very high (A in scenario h, i), the inventory level remains high for most of the time horizon and is sold at the very end only. Selling earlier would not result in high enough profits because the high price sensitivities would force to price too low. We observe that when the cross-price sensitivities are assigned in a scenario lower values than in another scenario, the equilibrium prices and production rates decrease, and profits decrease whether only one cross-sensitivity is lowered or both. This remark holds for symmetric and asymmetric scenarios, and with sensitivities increasing and decreasing with time. If only one of the cross-sensitivities is decreased, the effect is stronger on the supplier subject to the decrease. Moreover, comparing the asymmetric case with the symmetric case (a versus e, b versus g, c versus h, d versus j), we notice that supplier B's share of the total objective is greater when she has lower price sensitivities than A. It seems that this is due to the decrease in the sensitivity to her own price, although since comparing e and f, and h and j, we observe that her share of total revenues decreases when only her cross-sensitivity decreases.

7.2. Robust Formulation

We want to study the effect of the budget of uncertainty, so we will fix the parameters as given in the beginning of the previous section, and price sensitivities as in scenarios f and h. We will consider $\tilde{\alpha}(t)$ uncertain and take input parameters $\hat{\alpha}(\cdot)$ and $\Gamma(\cdot)$ that are linear functions of the time (although the linearity assumption is not necessary in general). To be able to isolate the effect of uncertainty, suppose that the parameters $\beta^{k,-k}(\cdot)$ are certain, i.e., $\hat{\beta}^{k,-k}(\cdot) = 0$. We choose for both suppliers $\hat{\alpha}(t) = b + at = 0.1 + 0.2t$. Indeed, it is reasonable to suppose that in practice, the inaccuracy of a forecast for the demand increases on the time horizon, i.e., that the length of the interval of feasible outcomes increases with time. This choice of inputs represents an uncertainty on the nominal value of parameter α of $\pm 2\%$ to $\pm 14\%$. We will consider the input parameters $\Gamma^k(\cdot)$ to be linear functions of the time and identical for the two suppliers: $\Gamma^k(t) = gt + c$, where $g, c \geq 0$, $g < 1$. We will compute the cumulative effective budget of uncertainty $\int_0^T \min\{t, \Gamma^k(t)\} dt$ as a measure of the global uncertainty in each scenario. See Table 1 for the values we consider. We now have to compute the corresponding inventory security levels $\Omega^A(\cdot)$, $\Omega^B(\cdot)$:

$$\Omega^k(t) = \begin{cases} (gt + c)(at + b) - \frac{a}{2}(gt + c)^2, & \frac{c}{1-g} < t \leq T, \\ \frac{a}{2}t^2 + bt, & 0 \leq t \leq \frac{c}{1-g}, \end{cases}$$

after calculations. In particular, $\Omega^k(0) = 0 \leq I^{k_0}$ and Assumption 2 is satisfied.

Table 1. Total objective value for various budgets of uncertainty.

	Scenario 1	Scenario 2	Scenario 3	Scenario 4	Scenario 5	Scenario 6
$\Gamma^k(t)$	$1 + 0.8t$	$1 + 0.5t$	$1 + 0.2t$	$0.5 + 0.8t$	$0.5 + 0.5t$	$0.5 + 0.2t$
$\int_0^T \min\{t, \Gamma^k(t)\} dt$	47.5	34	19.375	44.375	29.75	14.84375
Total objective value (scenario f)	1,008.5	1,010.3	1,013.8	1,008.8	1,011.0	1,014.8
Total objective value (scenario h)	1,253.5	1,257.2	1,264.9	1,253.6	1,258.4	1,266.3

In scenario f, the inventory levels must satisfy the minimum inventory security level, which is increasing with time. As a result, they decrease from the initial value (as in the deterministic case) until they reach and follow the minimum security level, and thus increase with time in the remaining of the time horizon. This effect is stronger as the budget of uncertainty increases because the minimum inventory level increases with the budget of uncertainty. To produce this higher inventory, the production rates increase with time faster than they do in the deterministic case, and both suppliers quickly use all the production capacity.

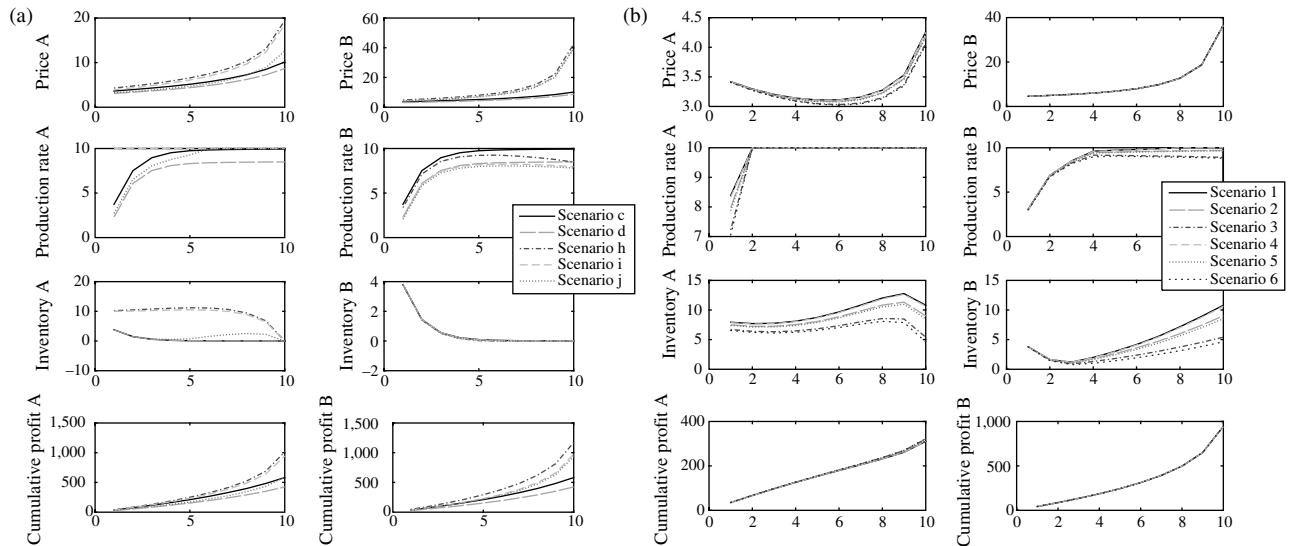
In scenario h (see Figure 4(b)), price sensitivities are decreasing with time. Supplier A has sensitivities twice bigger than supplier B. Supplier B's strategy is affected in the way explained before, i.e., guaranteeing the minimum inventory security levels forces her to have the inventory level that increases with time, thus to produce more than in the deterministic case. For supplier A, the minimum inventory security level affects her inventory level only at the last time period when instead of bringing it to zero, she brings it to the final minimum level. Because less sales will be allowed at the end of the time horizon (to meet the minimum level), supplier A sells more before; therefore, the inventory is kept at a level lower than in the deterministic case. To sell more, supplier A must produce more, and quickly reaches the production capacity, and as a result

must significantly decrease prices. Notice that supplier A realizes much lower profits than supplier B.

We observe moreover that as there is more uncertainty on the data, the inventory levels, production rates, and prices overall increase, but the profits decrease. Figure 5 shows that the sum of the suppliers' profits decreases as the cumulative effective budget of uncertainty increases, which illustrates the trade-off between performance (profit) and conservativeness (amount of uncertainty allowed). To illustrate that the robust formulation is beneficial when data is uncertain, we want to show that the nominal solution may yield infeasibility. We focus on scenarios f and h with the same inputs as in the previous numerical examples, and scenario 5 of budget of uncertainty. We generate 1,000 realizations of parameter $\tilde{\alpha}^k(\cdot)$ according to both (i) a uniform distribution on $[\alpha^k(\cdot) - \hat{\alpha}^k(\cdot), \alpha^k(\cdot) + \hat{\alpha}^k(\cdot)]$, and (ii) a normal distribution with mean $\alpha^k(\cdot)$ and standard deviation $0.5\hat{\alpha}^k(\cdot)$. The realized values are generated independently over time and for the two suppliers.

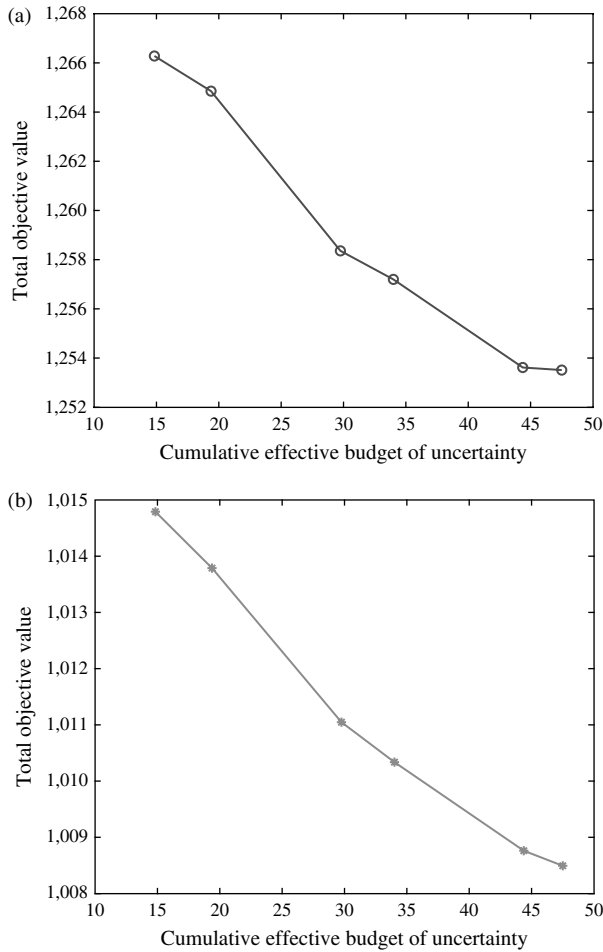
We apply the nominal solution to these realizations and record whether the no backorders constraint and upper bound on prices were violated. We calculate empirically the probability of violation of the constraints. We obtain that, under either distribution, the upper bound on prices is never violated. However, the no backorder constraint is violated very frequently. See Table 2 for the numerical results.

Figure 4. (a) Equilibrium in the case of price sensitivities decreasing with time (deterministic model) and (b) robust formulation: equilibrium in scenario h for various budgets of uncertainty.



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Figure 5. Results: Total objective value of the equilibrium as a function of the cumulative effective budget of uncertainty (a) under scenario h and (b) under scenario f.



These high probabilities of a violation of the no backorders constraint were expected because, in the deterministic case, often the inventory level was on the boundary (zero level), so when the data are slightly perturbed, the actual inventory level may easily become negative. The question of interest is to determine by how much this constraint is violated. To have an idea of the amplitude of violation of the no backorders constraint, we record the lowest value taken by the inventory level on the time horizon. The results are presented in the histograms in Figure 6. We observe that the inventory levels may reach values significantly below the zero level. As a result, it seems relevant to use robust optimization to avoid a situation where backorders are likely to occur at a significant level when there is uncertainty on the data.

As a comparison, we apply the robust solution (for each budget of uncertainty) to the generated simulations $\tilde{\alpha}(\cdot)$ and calculate the probability of constraint violations. We obtain that the upper bound on the price is never violated, and the probability that the no backorder constraint is violated is given in Table 3. Note that the robust solution is designed

Table 2. Probability of violation of the no backorders constraint for the nominal solution.

Distribution	Supplier A, scenario f	Supplier B, scenario f	Supplier A, scenario h	Supplier B, scenario h
Uniform (%)	83.3	82.9	88.1	100
Normal (%)	82.4	81.4	91.2	100

so that there is no violation for any realization that satisfies the bounds and the budget of uncertainty constraints. Because the generated realizations may not satisfy these constraints, it is possible that the no backorders constraint or the upper bound on prices are violated by the robust solution. We expect to verify that the higher the budget of uncertainty, the least likely it is to have violation of the no backorders constraint because the protection levels increase with the budget of uncertainty.

We also computed the average minimum inventory level attained over the time horizon for each robust solution and

Figure 6. Histogram of minimum inventory level reached in scenario h for (a) uniformly distributed realization and (b) normally distributed realization.

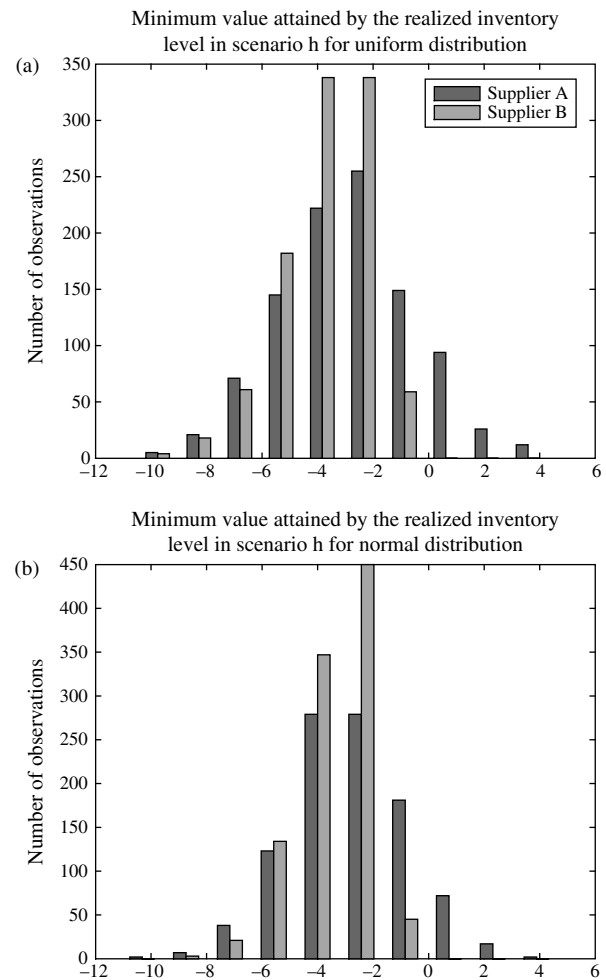


Table 3. Probability of violation of the no backorders constraint for the robust solution.

Budget of uncertainty scenario	Distribution	Supplier A, scenario f (%)	Supplier B, scenario f (%)	Supplier A, scenario h (%)	Supplier B, scenario h (%)
Scenario 1	Uniform	0	0	0	0
	Normal	0	0	0.1	0
Scenario 2	Uniform	0	0.8	1.3	0.2
	Normal	0	0.6	0.5	0
Scenario 3	Uniform	5.1	11.8	20.8	28.2
	Normal	1.8	5.0	15.2	21.0
Scenario 4	Uniform	0	0	0	0
	Normal	0	0	0.2	0
Scenario 5	Uniform	0	1.5	2.1	0.3
	Normal	0	1.0	0.6	0
Scenario 6	Uniform	10.6	21.6	28.1	48.9
	Normal	5.5	13.3	25.4	44.7

under both scenarios of price sensitivities. We obtain that the only case where the average is negative is supplier B, with scenario h of price sensitivities and scenario 6 of budget of uncertainty (which has the smallest cumulative effective value), when the realization is normally distributed. In that case, the average is -0.19 , which shows that the violation tends not to have a very large amplitude. In all other cases, the average was positive. We observe that the probability of violation of the no backorders constraint for the robust solutions is small in most cases, while it was very large for the nominal solution. Moreover, these probabilities are a decreasing function of the cumulative effective budget of uncertainty: the least overall uncertainty, the least protected the system against constraint violations, the higher the probabilities.

8. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at <http://orjournal.informs.org/>.

Acknowledgments

The authors thank the associate editor and the anonymous referees for their insightful comments, which have significantly improved this paper. This paper was supported in part by awards 0758061-CMII and 0556106-CMII from the National Science Foundation and the Singapore-MIT Alliance Program.

References

Adida, E., G. Perakis. 2006. A robust optimization approach to dynamic pricing and inventory control with no backorders. *Math. Programming Special Issue on Robust Optim.* **107**(1–2) 97–129.

Arrow, K. J., G. Debreu. 1954. Existence of an equilibrium for a competitive economy. *Econometrica* **22** 265–290.

Başar, T., G. J. Olsder. 1995. *Dynamic Noncooperative Game Theory*, 2nd ed. Academic Press, New York.

Ben-Tal, A., A. Nemirovski. 1998. Robust convex optimization. *Math. Oper. Res.* **23** 769–805.

Berridge, S., J. B. Krawczyk. 1997. Relaxation algorithms in finding Nash equilibria. Economics Working Paper Archive, Computational Economics Series, Washington University, St. Louis.

Bertsimas, D., M. Sim. 2004. The price of robustness. *Oper. Res.* **52**(1) 35–53.

Bertsimas, D., A. Thiele. 2006. A robust optimization approach to inventory theory. *Oper. Res.* **54**(1) 150–168.

Cavazzuti, E., M. Pappalardo, M. Passacantando. 2002. Nash equilibria, variational inequalities, and dynamical systems. *J. Optim. Theory Appl.* **114**(3) 491–506.

Chan, L. M. A., Z. J. M. Shen, D. Simchi-Levi, J. L. Swann. 2004. Coordination of pricing and inventory decisions: A survey and classification. D. Simchi-Levi, S. D. Wu, Z. J. M. Shen, eds. *Handbook of Quantitative Supply Chain Analysis: Modeling in the E-Business Era*, Chapter 9. International Series in Operations Research and Management Science. Kluwer Academic Publishers, New York, 335–392.

Cubiotti, P. 1997. Generalized quasi-variational inequalities in infinite-dimensional normed spaces. *J. Optim. Theory Appl.* **92**(3) 457–475.

Daniel, J. W. 1973. Stability of the solution of definite quadratic programs. *Math. Programming* **5** 41–53.

Dockner, E., S. Jørgensen, N. Van Long, G. Sorger. 2000. *Differential Games in Economics and Management Science*. Cambridge University Press, Cambridge, UK.

Elmaghraby, W., P. Keskinocak. 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Sci.* **49**(10) 1287–1309.

Haurie, A. 1995. Environmental coordination in dynamic oligopolistic markets. *Group Decision and Negotiation* **4**(1) 39–57.

Jørgensen, S., P. M. Kort, G. Zaccour. 1999. Production, inventory, and pricing under cost and demand learning effects. *Eur. J. Oper. Res.* **117** 382–395.

Mosco, U. 1976. Implicit variational problems and quasi-variational inequalities. A. Dold, B. Eckmann, eds. *Nonlinear Operators and the Calculus of Variations*. Lecture Notes in Mathematics, Vol. 543. Springer-Verlag, Brussels, 83–156.

Pang, J. S., M. Fukushima. 2005. Quasivariational inequalities, generalized Nash equilibria, and multi-leader-follower games. *Computational Management Sci.* **1** 21–56.

Rosen, J. B. 1965. Existence and uniqueness of equilibrium points for concave n -person games. *Econometrica* **33**(3) 520–534.

Talluri, K., G. van Ryzin. 2004. *The Theory and Practice of Revenue Management*. Springer, New York.

Uryas'ev, S., R. Y. Rubinstein. 1994. On relaxation algorithms in computation of noncooperative equilibria. *IEEE Trans. Automatic Control* **39**(6) 1263–1267.

Vives, X. 1999. *Oligopoly Pricing. Old Ideas and New Tools*. MIT Press, Cambridge, MA.