

I. The Algebra of Supply and Demand

- A. If the demand and supply curves are both linear, it is easy to get solutions for the conditions of equilibrium.

Demand: $P = A - BQ$ (A and B are positive, so -B is a negative slope.)

Supply: $P = C + DQ$ (A and C are vertical intercepts, D is positive slope.)

So these are 2 simultaneous equations. The solution values are:

$$Q^* = (A - C)/(B + D) \qquad P^* = (AD + BC)/(B + D)$$

- B. Now consider shifts in the determinants of equilibrium (comparative statics). In the algebra, this means shifts in A, B, C, or D.

1. Shift Demand Rightward (increase)

- A. Increase intercept A (shift curve upward) or decrease B (tilt about vertical intercept which makes the curve more horizontal).

$$Q^{**} = (A' - C)/(B + D) \quad \text{where } A' = A + \Delta A, \text{ and } \Delta A = \text{change in } A$$

Since B, D, A and ΔA are all positive, $Q^{**} > Q^*$

- B. Also, $P^{**} = (A'D + BC)/(B + D) > P^*$

2. Shift Supply Rightward (increase)

- A. Decrease intercept (C), downward movement of curve, or decrease D (tilt curve, more horizontal). $C' = C + \Delta C$, where ΔC is a negative increment.

$$Q^{***} = (A - C')/(B + D) > Q^*$$

- B. $P^{***} = (AD + BC')/(B + D) < P^*$

II. The Relationships Among Totals, Averages and Marginals

This holds for Total, Average and Marginal Utility, Total, Average and Marginal Cost, and more. This example is for a straight line demand curve with vertical and horizontal intercepts at 10.

Total: Revenue (R) \equiv Price (P) times Quantity (Q)

Average Revenue (AR) \equiv P \equiv R/Q (P is \$ per unit, R is \$)

Marginal Revenue (MR) \equiv limit ($\Delta R / \Delta Q$) (As $\Delta Q \rightarrow 0$)
Or, the Marginal is the slope of the Total curve.

| Q | P, or AR | R | MR |
|----|----------|----|----|
| 0 | 10 | 0 | -- |
| 1 | 9 | 9 | 9 |
| 2 | 8 | 16 | 7 |
| 3 | 7 | 21 | 5 |
| 4 | 6 | 24 | 3 |
| 5 | 5 | 25 | 1 |
| 6 | 4 | 24 | -1 |
| 7 | 3 | 21 | -3 |
| 8 | 2 | 16 | -5 |
| 9 | 1 | 9 | -7 |
| 10 | 0 | 0 | -9 |

6 Propositions on Total, Average, and Marginal Relationships

1. When a Total is rising, the corresponding Marginal is positive.
2. When a Total is falling, the corresponding Marginal is negative.
3. When a Total reaches a maximum, or a minimum, the Marginal is 0.
4. When the Average is falling, the Marginal must lie below it.
5. When the Average is rising, the Marginal must be above it.
6. When an Average is neither rising nor falling (at a min or max or other stationary point), the Marginal must equal the Average.

A. From an Average to a Marginal (Demand curve is an Average relationship, relating average cost to buyers (price) to quantity.)

1. If this is the demand relationship: $AR \equiv P = 15 - Q$
2. Multiply by Q to get the total paid (R): $R = PQ = 15Q - Q^2$
3. Differentiate R : $MR \equiv (dR/dQ) = 15 - 2Q$

4. Note that if AR is $f(Q)$, then R is $f(Q)Q$, the product of 2 functions, which we can differentiate to get the relationship:

$$MR \equiv (dR/dQ) = f(Q)(1) + (Q)f'(Q) \quad , \text{ and since } f(Q) \text{ is } AR,$$

$$MR - AR = MR - f(Q) = f(Q) + Qf'(Q) - f(Q) = Qf'(Q)$$

This furnishes the relationship in propositions 4, 5, and 6 above.

III. Ceteris Paribus (Other Things Equal)

A. Many applications: Utility -- CP gives indifference curves, changing goods amounts holding utility level equal; Budget line -- holding prices and income constant, changing goods amounts; and other functions with more than one variable.

B. Partial Differentiation

1. If $y = f(x_1, x_2, \dots, x_n)$, and the x variables are all independent of each other, so if x_1 changes it does not affect the others, then given a change in x_1 , Δx_1 , with x_2, \dots, x_n all remaining fixed, then the change in y will be Δy and the limit of $\Delta y/\Delta x_1$ as $\Delta x_1 \rightarrow 0$ will be the partial derivative of y with respect to x_1 . Partial derivative indicates that all the other independent variables in the function are held constant, and treated as constants in the process of differentiation. A partial derivative has a special symbol (not dy/dx anymore), a variant of the Greek δ , written $\partial y/\partial x_i$, and read "the partial derivative of y with respect to x_i ." Also, the partial derivative function is no longer $f'(x)$. Now the symbol would be f_1 , where the subscript indicates that x_1 alone is being allowed to vary.

2. If $y = f(x_1, x_2) = 3x_1^2 + x_1x_2 + 4x_2^2$ then to find $\partial y / \partial x_1$ we treat x_2 as a constant during differentiation. So the additive $4x_2^2$ will drop out, but terms remain if they are multiplicative constants, like x_1x_2 .

$$\frac{\partial y}{\partial x_1} = 6x_1 + x_2 \quad \text{And} \quad \frac{\partial y}{\partial x_2} = x_1 + 8x_2$$

Like the original function, the partial derivatives are functions of both x_1 and x_2 . If $(x_1, x_2) = (1, 3)$ then $f_1 = 9$ and $f_2 = 25$

3. The total derivative is the rate of change (slope) of a function of more than one x variable, when the x variables are not independent of each other. For example, $y = f(x, w)$ and $x = g(w)$, so that w affects y directly, through the function f , and indirectly, through the function g which alters x , and indirectly, through the effects of x on y . So f_w expresses only the direct effect, but a total derivative expresses both effects jointly. $(dy/dw) = f_x (dx/dw) + f_w (dw/dw)$ Note that $(dw/dw = 1)$.
4. For example, the total derivative, dy/dw , of $y = 3x - w^2$ where $x = g(w) = 2w^2 + w + 4$ is $(dy/dw) = 3(4w + 1) + (-2w) = 10w + 3$. An economic use is a utility function $U = U(c, s)$ where c is the amount of coffee consumed and s is the amount of sugar consumed. If sugar and coffee are complements and the function $s = g(c)$ indicates the relationship, then $U = U[c, g(c)]$. Then $(dU/dc) = \partial U / \partial c + [\partial U / \partial g(c)]g'(c)$

IV. Using Derivatives to Identify Maximum versus Minimum

- A. Maximum: First Order Condition, 1st derivative = 0; Second Order Condition, 2nd derivative < 0
- B. Minimum: FOC, 1st derivative = 0; SOC, 2nd derivative > 0

V. Lagrange Multipliers: Optimization Under Constraint

- A. Given the objective function $z = f(x, y)$ subject to the constraint that $g(x, y) = c$ where c is a constant, we can write the Lagrangian function as

$$Z = f(x, y) + \lambda[c - g(x, y)]$$

Z is considered a function of 3 variables: x, y, and λ . The Lagrangian multiplier, λ , is some unknown number. But if the constraint is satisfied, then the last term will vanish, whatever the value of λ . Then Z would be identical with z. To make λ vanish, we treat it as an additional variable. The FOC necessary conditions for a maximum are

$$Z_{\lambda} = c - g(x,y) = 0$$

$$Z_x = f_x - \lambda g_x = 0$$

$$Z_y = f_y - \lambda g_y = 0$$

The first condition guarantees that the constraint is satisfied, lambda vanishes, and the constrained maximum has been reached by treating it as an unconstrained maximization problem in 3 choice variables. These 3 conditions can be solved for the values of x and y that maximize z, given the constraint.

- B. A consumer has the utility function: $U = x_1x_2 + 2x_1$. The marginal utilities are U_1 and U_2 (the partial derivatives) and are positive for all positive levels of x_1 and x_2 . So for maximum utility, this consumer would buy infinite amounts of both goods, which is not a particularly useful result. If the consumer has \$60, and the price of x_1 is \$4 while the price of x_2 is \$2, then the consumer's purchasing power, measured by the budget constraint, is $4x_1 + 2x_2 = 60$. This lowers the range of x_1, x_2 values, and thus the range of the utility function, which is the "objective" function.

The Lagrangian function: $Z = x_1x_2 + 2x_1 + \lambda(60 - 4x_1 - 2x_2)$

The FOC are $\partial Z/\partial \lambda = 60 - 4x_1 - 2x_2 = 0$

$$\partial Z/\partial x_1 = x_2 + 2 - 4\lambda = 0$$

$$\partial Z/\partial x_2 = x_1 - 2\lambda = 0$$

This can be solved by substitution and elimination of variables to get $x_1 = 8$ and $x_2 = 14$. The utility of an income of 60 is 128.

- C. If the income amount and prices are not known, the first constraint from the Lagrangian above becomes $y - p_1x_1 - p_2x_2 = 0$, where y is the income and p_1 and p_2 are the prices. The second FOC becomes $x_2 + 2 - p_1\lambda = 0$.

The third FOC becomes $x_1 - p_2\lambda = 0$. These FOC can be solved for demand functions for goods x_1 and x_2 as functions of the prices and income by eliminating λ , then solving for one of the x 's and substituting that value in the first FOC.

$$\text{Demand for } x_1: x_1 = (y + 2p_2)/2p_1$$

$$\text{Demand for } x_2: x_2 = (y - 2p_2)/2p_2$$

D. These functions can be differentiated and solved for the price and income elasticities.

1. For the own price elasticity of x_1 , $\epsilon = (\partial x_1 / \partial p_1)(p_1/x_1)$. Substitute the demand function for x_1 after differentiating and cancel terms. So ϵ_{11} (the price elasticity of x_1 with respect to p_1) = $[-(y + 2p_2) p_1 / 2p_1^2] / [(y + 2p_2) / 2p_1] = -1$.
2. For the cross price elasticity of x_1 , $\epsilon = (\partial x_1 / \partial p_2)(p_2/x_1)$. Same process, just differentiate x_1 with respect to p_2 and substitute in the demand function for x_1 .
3. For the income elasticity of x_1 , $\eta = (\partial x_1 / \partial y)(y/x_1)$. Substitute the demand function in for x_1 after differentiating and cancel terms.

I. Present Values: Time Compounding and Discounting

The price of the right to a stream of payments, or the price today of an asset to be delivered in the future, is the present value.

For a payment in one year,

$P = R_1 / (1 + i)$, where P is the present value, R_1 is the principal plus yield coming all at once one year from now, and i is the interest rate that could be earned on the sum P used to buy the stream. So $(1 + i)P = R_1$, or if you invest P you get back P itself plus iP at the end of the year.

For a stream of annual returns lasting forever,

$P = R/i$ where R (no subscript) is the annual return repeated into the future.

The general discounting formula for any stream of income R_1 in the first year, R_2 in the second, and so on for N years is

$$P = [R_1 / (1 + i)] + [R_2 / (1 + i)^2] + [R_3 / (1 + i)^3] + \dots + [R_N / (1 + i)^N]$$

If all the R 's are equal, the general discounting formula becomes the annuity formula

$$P = (R/i)\{1 - [1/(1 + i)^N]\}$$

II. Algebra of Growth Rates

The relationships of growth in component parts to the growth of the whole is easily calculated.

For multiplicative relationships, such as Cobb-Douglas functions, $Y = AK^a L^b$, the growth rate of the whole is the sum of the growth rates of the parts, multiplied by the exponents (minus signs for division):

$$Y\% \Delta = A\% \Delta + aK\% \Delta + bL\% \Delta$$

If the relationship is additive, $Y = K + L$, the growth rate of the whole is the growth rates of the parts, each weighted by that part's share of the total:

$$Y\% \Delta = [K/(K + L)]K\% \Delta + [L/(K + L)]L\% \Delta$$

