Time evolution of field operators

The relation between Schroedinger picture and Heisenberg picture is

$$|\Psi(t)\rangle_S = e^{-iHt}|\Psi(0)\rangle_S = e^{-iHt}|\Psi\rangle_H \ , \quad \text{therefore} \quad \hat{O}_H(t) = e^{iHt}\hat{O}_S e^{-iHt}$$

Expanding for a small time elapsed, we obtain the time differential equation in the Heisenberg picture,

$$d\hat{O}_H(t)/dt = i[H, \hat{O}_H(t)] \ .$$

The sign was wrongly given in the lecture on March 2, 2005. Please correct it. To simplify our notation, we are going to suppress the Heisenberg subscript $_H$ and remove the symbol "hat" ^ which is placed over an operator.

In field theory, we can consider that the space variable x is merely a continuous label of the field ϕ, just like the index j a label of the q_j in mechanics. The quantization procedure of fields is called second quantization, which is widely used not only in the high energy physics, but also in the condensed matter physics. The parallel relations are

$$p_j = \frac{\partial L}{\partial \dot{q}_j} \quad \Longrightarrow \quad \pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)}$$

$$[q_k, p_j] = i\delta_{jk} \quad \Longrightarrow \quad [\phi(t,\boldsymbol{y}), \pi(t,\boldsymbol{x})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \ .$$

For the simple example of a charged free scalar field $\phi(x)$, we have

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - m^2\phi^*\phi = \dot{\phi}^*\dot{\phi} - \boldsymbol{\nabla}\phi^*\cdot\boldsymbol{\nabla}\phi - m^2\phi^*\phi$$

$$H = \int d^3\boldsymbol{x}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\dot{\phi} + \frac{\partial\mathcal{L}}{\partial\dot{\phi}^*}\dot{\phi}^* - \mathcal{L}\right) = \int d^3\boldsymbol{x}\left(\dot{\phi}^*\dot{\phi} + \boldsymbol{\nabla}\phi^*\cdot\boldsymbol{\nabla}\phi + m^2\phi^*\phi\right)$$

Note that it is positive definite in the form. The momentum density w.r.t. ϕ is $\pi = \dot{\phi}^*$. Similarly, The momentum density w.r.t. ϕ^* is $\pi^* = \dot{\phi}$. Therefore,

$$[\phi(t,\boldsymbol{y}), \dot{\phi}^*(t,\boldsymbol{x})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \ , \quad [\phi^*(t,\boldsymbol{y}), \dot{\phi}(t,\boldsymbol{x})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y}) \ .$$

Otherwise, we have zero commutation relations

$$[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})] = 0 \ , \ [\phi(t,\boldsymbol{x}), \phi^*(t,\boldsymbol{y})] = 0 \ ,$$

$$[\dot{\phi}^*(t,\boldsymbol{x}), \dot{\phi}(t,\boldsymbol{y})] = 0 \ , \ [\dot{\phi}(t,\boldsymbol{x}), \dot{\phi}(t,\boldsymbol{y})] = 0 \ , [\dot{\phi}(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})] = 0 \ ,$$

etc. First we check the Heisenberg relation, $\dot{\phi}(t,\boldsymbol{x}) = i[H, \phi(t,\boldsymbol{x})]$. It is trivially satisfied because

$$\text{RHS} = i[H, \phi(t,\boldsymbol{x})] = i\int d^3\boldsymbol{y}\left[\dot{\phi}^*(t,\boldsymbol{y})\dot{\phi}(t,\boldsymbol{y}), \phi(t,\boldsymbol{x})\right] = i\int d^3\boldsymbol{y}\left[\dot{\phi}^*(t,\boldsymbol{y}), \phi(t,\boldsymbol{x})\right]\dot{\phi}(t,\boldsymbol{y})$$

$$= \int d^3\boldsymbol{y}\delta^3(\boldsymbol{x}-\boldsymbol{y})\dot{\phi}(t,\boldsymbol{y}) = \dot{\phi}(t,\boldsymbol{x}) = \text{LHS}$$

This only provides a consistent relation, but not any new information. Let us repeat this procedure on $\dot{\phi}(t,\boldsymbol{x})$, we expect $\ddot{\phi}(t,\boldsymbol{x}) = i[H, \dot{\phi}(t,\boldsymbol{x})]$

$$i[H, \dot{\phi}(t,\boldsymbol{x})] = i\int d^3\boldsymbol{y}\left[\dot{\phi}^*(\boldsymbol{y})\dot{\phi}(\boldsymbol{y}) + \boldsymbol{\nabla}\phi^*(\boldsymbol{y})\cdot\boldsymbol{\nabla}\phi(\boldsymbol{y}) + m^2\phi^*(\boldsymbol{y})\phi(\boldsymbol{y}), \dot{\phi}(\boldsymbol{x})\right]$$

We have suppressed the common time t variable. Note that the first term vanishes.

$$i\int d^3\boldsymbol{y}\left[\boldsymbol{\nabla}\phi^*(\boldsymbol{y})\cdot\boldsymbol{\nabla}\phi(\boldsymbol{y}),\dot{\phi}(\boldsymbol{x})\right]=i\int d^3\boldsymbol{y}\left[\boldsymbol{\nabla}\phi^*(\boldsymbol{y}),\dot{\phi}(\boldsymbol{x})\right]\cdot\boldsymbol{\nabla}\phi(\boldsymbol{y})$$

$$=-\int d^3\boldsymbol{y}(\boldsymbol{\nabla}\delta^3(\boldsymbol{x}-\boldsymbol{y}))\cdot\boldsymbol{\nabla}\phi(\boldsymbol{y})=\nabla^2\phi(\boldsymbol{x})\ .$$

Note that the gradient always acts on the variable $\mathbf{y}$ except on the last step, it acts on $\mathbf{x}$. Let us look at the remaining piece,

$$i\int d^3\boldsymbol{y}\left[m^2\phi^*(\boldsymbol{y})\phi(\boldsymbol{y}),\dot{\phi}(\boldsymbol{x})\right]=i\int d^3\boldsymbol{y}\left[m^2\phi^*(\boldsymbol{y}),\dot{\phi}(\boldsymbol{x})\right]\phi(\boldsymbol{y})$$

$$=-m^2\int d^3\boldsymbol{y}\delta^3(\boldsymbol{x}-\boldsymbol{y})\phi(\boldsymbol{y})=-m^2\phi(\boldsymbol{x})$$

Combining all terms, we derive the Klein-Gordon equation for a relativistic boson field,

$$\ddot{\phi}-\nabla^2\phi+m^2\phi=0$$

It is worthwhile to point out that we use the product rule of commutators,

$$[AB,C]=ABC-CAB=A[B,C]+[A,C]B\ .$$

However, the structure in the above proof is not much altered with another rule,

$$[AB,C]=ABC-CAB=ABC+ACB-ACB-CAB=A\{B,C\}-\{C,A\}B$$

where the anti-commutator is defined as $\{A,B\}=AB+BA$. It is easy to show that the same equation of motion is obtained with the choice of anti-commutation relation, $\{\phi(t,\boldsymbol{y}),\pi(t,\boldsymbol{x})\}=i\delta^3(\boldsymbol{x}-\boldsymbol{y})$. The two different choices correspond to two different statistics, one for the fermion and the other one for the boson. Pauli has shown that causality and relativity mandate that the integer spin particles obey the boson statistics, and the half-integer spin particles obey the Fermi statistics.

Homework, extension to Dirac Fermion

Determine the Noether current associated with the phase invariance $\psi\to\psi'=e^{i\alpha}\psi$ for a free Dirac field.

Following the example of the translational invariance of the charged scalar boson field, derive the energy-momentum tensor for a free Dirac field. Find the conserved Hamiltonian as a spatial integral.

Using the anti-commutation relation $\{\phi(t,\boldsymbol{y}),\pi(t,\boldsymbol{x})\}=i\delta^3(\boldsymbol{x}-\boldsymbol{y})$, show that the Heisenberg evolution equation $d\psi/dt=i[H,\psi]$ is equivalent to the Dirac equation.