Second quantized Schroedinger Equation

Find the Lagnagian density $\mathcal{L}(\psi^\dagger, \psi, \mathbf{r})$ such that the Euler equations

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} + \sum_{i=1,2,3} \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

implies the Schroedinger equation.

$$i\partial_t \psi = -\frac{\nabla^2}{2m}\psi + V(\mathbf{r})\psi \ .$$

The system posses a symmetry of phase invaraince, $\psi \to e^{i\alpha}\psi$. Find the associated conserved current of this symmetry.

Find the conjugated momentum density $\pi(\mathbf{r}) = \partial\mathcal{L}/\partial\dot{\psi}$ and derive the Hamiltonian of the second quantized field.

Using either the commutation relation $[\psi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$ for the boson field, or the anti-commutation relation $\{\psi(\mathbf{x}), \pi(\mathbf{y})\} = i\delta^3(\mathbf{x} - \mathbf{y})$ for the fermion field, show that the equation motion is consistent with the Heisenberg picture, $\dot{\psi} = i[H, \psi]$.

Expand ψ in terms of the basis eigen-fucntions $\phi_i(\mathbf{r})$,

$$\left(-\frac{\nabla^2}{2m} + V(\mathbf{r})\right)\phi_k(\mathbf{r}) = E_k\phi_k(\mathbf{r}) \ .$$

$$\psi(\mathbf{x}, t) = \sum_k a_i \phi_i(\mathbf{r}) e^{-iE_k t}$$

Find the relation $[a_k, a_m^\dagger]$ for the boson, or $\{a_k, a_m^\dagger\}$ for the fermion.