# Homework, extension to Dirac Fermion

Determine the Noether current associated with the phase invariance $\psi \to \psi' = e^{i\alpha}\psi$ for a free Dirac field.

Following the example of the translational invariance of the charged scalar boson field, derive the energy-momentum tensor for a free Dirac field. Find the conserved Hamiltonian as a spatial integral.

Using the anti-commutation relation $\{\phi(t,\boldsymbol{y}),\pi(t,\boldsymbol{x})\} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})$, show that the Heisenberg evolution equation $d\psi/dt = i[H,\psi]$ is equivalent to the Dirac equation.

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$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi$$

$\psi \to \psi' = e^{i\alpha}\psi$ invariance implies

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\Delta\psi \ , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = i\bar{\psi}\gamma^\mu \ , \quad \Delta\psi = i\psi \ , \quad J^\mu = -\bar{\psi}\gamma^\mu\psi$$

The sign as an overall constant can be removed.

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$$T^\mu_{\ \beta} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)}\partial_\beta \psi - g^\mu_\beta \mathcal{L} \ .$$

$$T^\mu_{\ \beta} = \bar{\psi} i\gamma^\mu \partial_\beta \psi - \bar{\psi} i\gamma^\mu \partial_\mu \psi \delta^\mu_\beta + m\bar{\psi}\psi\delta^\mu_\beta$$

$$T^0_{\ 0} = \bar{\psi}(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi = \mathcal{H} \ , \quad H = \int d^3\boldsymbol{x}\mathcal{H} \ , \text{ as the conserved energy.}$$

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$$i[H,\psi(\mathbf{y})] = i\int d^3\boldsymbol{x}[\bar{\psi}(\mathrm{x})(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi(\mathrm{x}), \psi(\mathbf{y})]$$

The canonical momentum density $\pi$ conjugated to $\psi$ is $\partial\mathcal{L}/\partial(\dot{\psi}) = \bar{\psi}i\gamma^0 = i\psi^\dagger$. The anticommutation relation becomes

$$\{\psi_a(\boldsymbol{y}), \psi_b^\dagger(\boldsymbol{x})\} = \delta_{ab}\delta^3(\mathbf{x}-\mathbf{y}) \ , \quad \{\psi_a(\boldsymbol{y}), \psi_b(\boldsymbol{x})\} = 0 \ .$$

As $[AB,C] = A\{B,C\} - \{C,A\}B$,

$$i[H,\psi(\mathbf{y})] = -i\int d^3\boldsymbol{x}\{\psi(\mathbf{y}),\psi^\dagger(\mathrm{x})\}\gamma^0(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi(\mathrm{x}),$$

$$\dot{\psi}(\mathbf{y}) = -i\gamma^0(-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi(\mathbf{y}),$$

$$i\gamma^0\partial_0\psi(\mathbf{y}) = (-i\boldsymbol{\gamma}\cdot\boldsymbol{\nabla} + m)\psi(\mathbf{y}), \qquad (i\gamma^\mu\partial_\mu - m)\psi = 0$$

The Dirac equation is recovered.