Second quantized Schroedinger Equation

Find the Lagnagian density $\mathcal{L}(\psi^\dagger, \psi, \mathbf{r})$ such that the Euler equations

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)} + \sum_{i=1,2,3} \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0$$

implies the Schroedinger equation.

$$i\partial_t \psi = -\frac{\nabla^2}{2m}\psi + V(\mathbf{r})\psi \ .$$

Try

$$\mathcal{L}(\psi^\dagger, \psi, \mathbf{r}) = i\psi^* \partial_t \psi - \frac{1}{2m}(\boldsymbol{\nabla}\psi^*) \cdot (\boldsymbol{\nabla}\psi) - \psi^* V(\mathbf{r})\psi$$

We *pretend* that ψ and ψ^* are independent variables. From the Euler equation on ψ, we obtain

$$\partial_t(i\psi^*) - \frac{1}{2m}\nabla^2\psi^* + V(\mathbf{r})\psi^* = 0$$

The complex conjugate is the usual form of the Schroedinger equation. The Euler equation on ψ^* get the same result with less algebra.

The system posses a symmetry of phase invaraince, $\psi \to e^{i\alpha}\psi$. Find the associated conserved current of this symmetry.

$\Delta\psi = i\psi$, $\Delta\psi^* = -i\psi^*$

$$J^0 = \frac{\partial \mathcal{L}}{\partial(\partial_t \psi)}\Delta\psi \ , \quad J^k = \frac{\partial \mathcal{L}}{\partial(\partial_k \psi)}\Delta\psi + \frac{\partial \mathcal{L}}{\partial(\partial_k \psi^*)}\Delta\psi^*$$

$$J^0 = -\psi^*\psi \ , \quad J^k = -i\frac{1}{2m}(\partial_k\psi^*)\psi + i\frac{1}{2m}\psi^*\partial_k\psi$$

This is the usual probability current with a sign flip.

Find the conjugated momentum density $\pi(\mathbf{r}) = \partial\mathcal{L}/\partial\dot{\psi}$ and derive the Hamiltonian of the second quantized field.

The momentum w.r.t. ϕ is $\Pi = i\psi^*$, and there is no momentum w.r.t. ψ^*.

$$\mathcal{H} = \Pi\dot{\psi} - \mathcal{L} = \frac{1}{2m}(\boldsymbol{\nabla}\psi^*)(\boldsymbol{\nabla}\psi) + V\psi^*\psi$$

Using either the commutation relation $[\psi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})$ for the boson field, or the anti-commutation relation $\{\psi(\mathbf{x}), \pi(\mathbf{y})\} = i\delta^3(\mathbf{x} - \mathbf{y})$ for the fermion field, show that the equation motion is consistent with the Heisenberg picture, $\dot{\psi} = i[H, \psi]$. This step involves a lot of delta functions and algebra. It is not repeated here.

Expand ψ in terms of the basis eigen-fucntions $\phi_i(\mathbf{r})$,

$$\left(-\frac{\nabla^2}{2m} + V(\mathbf{r})\right)\phi_k(\mathbf{r}) = E_k\phi_k(\mathbf{r}) \ .$$

$$\psi(\mathbf{x}, t) = \sum_k a_i\phi_i(\mathbf{r})e^{-iE_kt}$$

Find the relation $[a_k, a_m^\dagger]$ for the boson, or $\{a_k, a_m^\dagger\}$ for the fermion.

The basis expansion gives

$$a_i = \int \phi_i^*(\mathbf{r})\psi(\mathbf{r}, 0)d^3\mathbf{r} \ , \quad a_k^\dagger = \int \phi_k(\mathbf{r}')\psi^\dagger(\mathbf{r}', 0)d^3\mathbf{r}'$$

Let us work out the bosonic case first. The fermionic case also follows.

$$[a_i, a_k^\dagger] = \int d^3\mathbf{r}d^3\mathbf{r}'\phi_i^*(\mathbf{r})\phi_k(\mathbf{r}')[\psi(\mathbf{r}, 0), \psi^\dagger(\mathbf{r}', 0)]$$

As $[\psi(\mathbf{r}, 0), \psi^\dagger(\mathbf{r}', 0)] = \delta^3(\mathbf{r} - \mathbf{r}')$, and $\int d^3\mathbf{r}\phi_i^*(\mathbf{r})\phi_k(\mathbf{r}) = \delta_{ik}$, we obtain the desired relations.