Helicity amplitude

The two-component left-handed Weyl spinor for a fermion, denoted by $\mathbf{u_p}$ satisfies

$$(p \cdot \bar{\sigma})\mathbf{u_p} = 0 \quad \text{with} \quad p = (E_\mathbf{p}, \mathbf{p}) \ .$$

$E_\mathbf{p} = |\mathbf{p}|$ in the massless limit. If $\mathbf{p} = E\hat{\mathbf{z}}$, the particle is moving along the $+z$ axis.

$$E(1+\sigma_z)\mathbf{u}_{+E\mathbf{z}} = 0 \ , \quad \mathbf{u}_{+E\mathbf{z}} = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The normalization $\sqrt{2E}$ follows the relativistic phase space convention.

Now for an antifermion, $\mathbf{v_p}$ of physical energy and momentum, $p = (E_\mathbf{p}, \mathbf{p})$, the mathematical frequency and wave-number is $-p$. So the equation is

$$(-p \cdot \bar{\sigma})\mathbf{v_p} = 0 \quad \text{with} \qquad .$$

If $\mathbf{p} = E\hat{\mathbf{z}}$, the anti-particle is moving along the $+z$ axis.

$$E(1+\sigma_z)\mathbf{v}_{+E\mathbf{z}} = 0 \ , \quad \mathbf{v_p} = \sqrt{2E}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

It is the same as in the case of a fermion.

In a pair production, fermion and antifermion are traveling in opposite direction in the CM frame. So for an antifermion along $-z$ axis,

$$\mathbf{v}_{-E\mathbf{z}} = \sqrt{2E}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To avoid long expressions, sometimes we suppress the tedious subscripts. The reader must understand the context of symbols. The vertex amplitude involves

$$\mathbf{u}^\dagger \bar{\sigma}^\mu \mathbf{v} = \bar{u}\gamma^\mu \tfrac{1}{2}(1-\gamma_5)v = 2E(\mathbf{x} + i\mathbf{y})$$

The first equality relates the Weyl structure to the Dirac structure. The second equality is obtained by brute force. There is no time component because of the current conservation. Transverse nature is reflected by the absence of the z component. When it is contracted with a source polarization $\epsilon_- = \frac{1}{\sqrt{2}}(\mathbf{x} - i\mathbf{y})$, we get the maximum contraction

$$\mathcal{M}_L = \epsilon_- \cdot \bar{u}\gamma^\mu \tfrac{1}{2}(1-\gamma_5)v \quad \longrightarrow \quad -2\sqrt{2}E$$

The pair can be ejected at any polar angle θ with respect to the z-axis. Then $\mathbf{x}$ becomes $\cos\theta\mathbf{x} - \sin\theta\mathbf{z}$,

$$[\mathcal{M}_L]_\theta \quad \longrightarrow \quad -(1+\cos\theta)\sqrt{2}E$$

The eject pair has an interesting $(1-\cos\theta)^2$ distribution which is forward backward asymmetry. If the fermion field is of the right-handed chirality, then

$$[\mathcal{M}_R]_\theta \quad \longrightarrow \quad -(1-\cos\theta)\sqrt{2}E$$

The case of $e^-e^+ \to f\bar{f}$ with the left-handed chirality is a simple example of the above source polarization $\epsilon_- = \bar{v}(e^+)\gamma^\mu \frac{1}{2}(1-\gamma_5)u(e^-)$,

$$\mathcal{A}_{LL} = \bar{v}(e^+)\gamma^\mu \tfrac{1}{2}(1-\gamma_5)u(e^-)\ \bar{u}(f)\gamma^\mu \tfrac{1}{2}(1-\gamma_5)v(\bar{f})$$
$$= 4E^2(1+\cos\theta) = 4(p_{e+}\cdot p_f) \text{ or } 4(p_{e-}\cdot p_{\bar{f}})$$
$$|\mathcal{A}_{LL}|^2 = |\bar{v}(e^+)\gamma^\mu \tfrac{1}{2}(1-\gamma_5)u(e^-)\bar{u}(f)\gamma^\mu \tfrac{1}{2}(1-\gamma_5)v(\bar{f})|^2$$
$$= 16(p_{e+}\cdot p_f)(p_{e-}\cdot p_{\bar{f}}) = 4u^2$$

We have used the well adopted Mandelstam kinematic variables, $s=(p_{e^+}+p_{e^-})^2$, $t=(p_{e^-}-p_f)^2$, and $u=(p_{e^-}-p_{\bar{f}})^2$.

For different chirality combinations,

$$|\mathcal{A}_{RR}|^2 = 4u^2\ ,\quad |\mathcal{A}_{LR}|^2 = 4t^2\ ,\quad |\mathcal{A}_{RL}|^2 = 4t^2\ .$$

Summing up all helicities and including Q_f and the virtual photon propagator, we find

$$\sum |\mathcal{M}|^2 = N_c Q_f^2 e^8 (t^2+u^2)/s^2$$

For the muon production, $Q_\mu = -1$. For quarks, $Q_u = \frac{2}{3}$, $Q_d = -\frac{1}{3}$, $Q_s = -\frac{1}{3}$, etc. Finally, including the phase space integrations, we find the cross section,

$$d\sigma = (\text{spin ave.})\,\frac{1}{|v_e - v_{e^+}|2E_e 2E_{e^+}} \sum_{\text{spin}} |\mathcal{M}|^2 d_2(\text{PS})$$

Where

$$\begin{aligned} d_2(\text{PS}) &= (2\pi)^4\delta^4(\textstyle\sum p)\frac{d^3\vec{\mu}}{(2\pi)^3 2\mu^0}\frac{d^3\vec{\mu}^+}{(2\pi)^3 2\mu^{+0}} \\ &= \frac{1}{8\pi}\left(\frac{2|\vec{p}_\mu|}{\sqrt{s}}\right)_{\text{CM}}\left(\frac{d\Omega_{\text{CM}}}{4\pi}\right) \end{aligned}$$

$$d\sigma = \frac{1}{8s}8e^4\frac{t^2+u^2}{s^2}\left(\frac{1}{8\pi}\right)\left(\frac{d\Omega_{\text{CM}}}{4\pi}\right)$$

$$\begin{aligned} d\sigma &= \frac{1}{8s}8e^4\frac{1}{4}\left[(1-\cos\theta)^2+(1+\cos\theta)^2\right]\left(\frac{1}{8\pi}\right)\left(\frac{d\Omega_{\text{CM}}}{4\pi}\right) \\ &= \frac{\alpha^2\pi}{s}(1+\cos\theta^2)\left(\frac{d\Omega}{4\pi}\right) \\ \sigma &= \frac{\alpha^2\pi}{s}\left(1+\frac{1}{3}\right) \end{aligned}$$