2-d Angular Momentum

It is useful to use polar coordinates,

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$\frac{\partial}{\partial\phi} = \frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial\phi} = r\frac{\partial\cos\phi}{\partial\phi}\frac{\partial}{\partial x} + r\frac{\partial\sin\phi}{\partial\phi}\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial\phi} = -r\sin\phi\frac{\partial}{\partial x} + r\cos\phi\frac{\partial}{\partial y}$$

$$\frac{\partial}{\partial\phi} = +x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$$

$$\left(-i\hbar\frac{\partial}{\partial\phi}\right) = +x\left(-i\hbar\frac{\partial}{\partial y}\right) - y\left(-i\hbar\frac{\partial}{\partial x}\right) = xp_x - yp_x = L_z$$

$$\boxed{L_z = -i\hbar\partial/\partial\phi}$$

Let a definite angular momentum state be $\Psi(\phi)$ as a function of ϕ, $L_z\Psi(\phi) = m\hbar\Psi(\phi)$, with angular momentum value be $m\hbar$, measured in unit of $\hbar$. Note that $\hbar$ carries the dimension of angular momentum), thus the value m is dimensionless.

$$-i\frac{d}{d\phi}\Psi = m\Psi \ , \quad \Psi(\phi) = e^{im\phi}$$

As the wave function has to be single-valued when go around 2π, m has to be integer. This proves that the angular momentum must be quantized as $m\hbar$, where $m = 0, \pm1, \pm2, \pm3, \cdots$.

Three Dimensional Angular Momentum

Similar algebra, though more sophisticated, can show that

$$L^2 = \mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2 \ , \text{ quantized as } \ell(\ell+1)\hbar^2, \text{ for } \ell = 0, 1, 2, 3, \cdots.$$

and $L_z = -i\hbar\partial/\partial\phi$ in the spherical coordinate. The component' L_z is quantized to be $m\hbar$ as before. $|m| = 0, 1, 2, 3, \cdots, \ell$.