make in one lifetime trips that would require hundreds of years as viewed from earth. Since this requires rockets that travel very close to the speed of light, it is not likely to happen soon! See Problem 1.22 for further discussion of this effect.

Lengths Perpendicular to the Relative Motion

We have so far discussed lengths that are parallel to the relative velocity, such as the length of a train in its direction of motion. What happens to lengths perpendicular to the relative velocity, such as the height of the train? It is fairly easy to show that for such lengths, there is no contraction or expansion. To see this, consider two observers, Q at rest in S and Q' at rest in S', and suppose that Q and Q' are equally tall when at rest. Now, let us assume for a moment that there is a contraction of heights analogous to the length contraction (1.29). If this is so, then as seen by Q, observer Q' will be shorter as he rushes by. We can test this hypothesis by having Q' hold up a sharp knife exactly level with the top of his head; if Q' is shorter, Q will find himself scalped (or worse) as the knife goes by.

This experiment is completely symmetric between the two frames S and S': There is one observer at rest in each frame, and the only difference is the direction in which each sees the other moving.* Therefore, it must also be true that as seen by Q', it is Q who is shorter. But this implies that the knife will miss Q. Since it cannot be true that Q is both scalped and not scalped, we have arrived at a contradiction, and there can be no contraction. By a similar argument, there can be no expansion, and, in fact, the knife held by Q' simply grazes past Q's scalp, as seen in either frame. We conclude that lengths perpendicular to the relative motion are unchanged; and the Lorentz-contraction formula (1.29) applies only to lengths parallel to the relative motion.

1.11 The Lorentz Transformation

We are now ready to answer an important general question: If we know the coordinates x, y, z, and time t of an event, as measured in a frame S, how can we find the coordinates x', y', z', and t' of the same event as measured in a second frame S'? Before we derive the correct relativistic answer to this question, we examine briefly the classical answer.

We consider our usual two frames, S anchored to the ground and S' anchored to a train traveling with velocity $\mathbf{v}$ relative to S, as shown in Fig. 1.8. Because the laws of physics are all independent of our choice of origin and orientation, we are free to choose both axes Ox and $O'x'$ along the same line, parallel to $\mathbf{v}$, as shown. We can further choose the origins of time so that $t = t' = 0$ at the moment when O' passes O. We will sometimes refer to this arrangement of systems S and S' as the **standard configuration**.

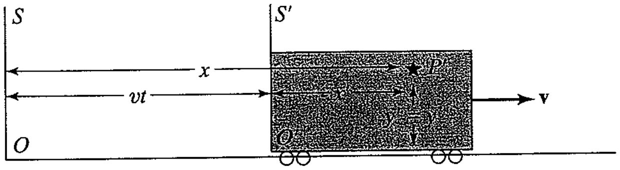

FIGURE 1.8

In classical physics the coordinates of an event are related as shown.

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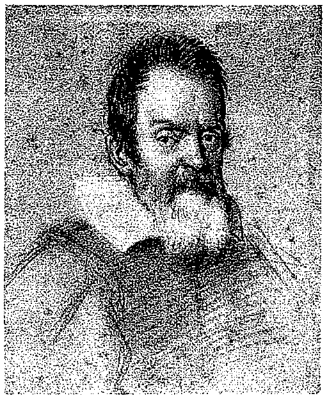
*Note that our previous two thought experiments were asymmetric, requiring two observers in one of the frames, but only one in the other.

Galileo Galilei

(1564–1642, Italian)



Considered by many the father of modern science, Galileo understood the importance of experiment and theory and was a master of both. Although he did not discover the telescope, he improved it and was the first to use it as a tool of astronomy, discovering the mountains on the moon, phases of Venus, moons of Jupiter, stars of the Milky Way, and sunspots and rotation of the sun. Among his many contributions to mechanics, he established the law of inertia and proved that gravity accelerates all bodies equally and that the period of a small-amplitude pendulum is independent of the amplitude. He understood clearly that the laws of mechanics hold in all unaccelerated frames, arguing that inside an enclosed cabin it would be impossible to detect the uniform motion of a ship. This argument appeared in his *Dialogue on the Two Chief World Systems* and was used to show that the earth could perfectly well be moving in orbit around the sun without our being aware of it in everyday life. For publishing this book, he was found guilty of heresy by the Holy Office of the Inquisition, and his book was placed on the Index of Prohibited Books — from which it was not removed until 1835.

Now consider an event, such as the explosion of a small firecracker, that occurs at position x, y, z, and time t as measured in S. Our problem is to calculate, in terms of x, y, z, t, (and the velocity v) the coordinates x', y', z', t' of the same event, as measured in S' — accepting at first the classical ideas of space and time. First, since time is a universal quantity in classical physics, we know that $t' = t$. Next, from Fig. 1.8 it is easily seen that $x' = x - vt$ and $y' = y$ (and, similarly, $z' = z$, although the z coordinate is not shown in the figure). Thus, according to the ideas of classical physics,

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t \end{aligned} \tag{1.31}$$

These four equations are often called the **Galilean transformation** after Galileo Galilei, who was the first person known to have considered the invariance of the laws of motion under this change of coordinates. They transform the coordinates x, y, z, t of any event as observed in S into the corresponding coordinates x', y', z', t' as observed in S'.

If we had been given the coordinates x', y', z', t' and wanted to find x, y, z, t, we could solve the equations (1.31) to give

$$\begin{aligned} x &= x' + vt' \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned} \tag{1.32}$$

Notice that the equations (1.32) can be obtained directly from (1.31) by exchanging x, y, z, t with x', y', z', t' and replacing v by $-v$. This is because the relation of S to S' is the same as that of S' to S except for a change in the sign of the relative velocity.

The Galilean transformation (1.31) cannot be the correct relativistic relation between x, y, z, t, and x', y', z', t'. (For instance, we know from time dilation that the equation $t' = t$ cannot possibly be correct.) On the other hand the Galilean transformation agrees perfectly with our everyday experience and so must be correct (to an excellent approximation) when the speed v is small compared to c. Thus the correct relation between x, y, z, t and x', y', z', t' will have to reduce to the Galilean relation (1.31) when v/c is small.

To find the correct relation between x, y, z, t and x', y', z', t', we consider the same experiment as before, which is shown again in Fig. 1.9. We have noted before that distances perpendicular to $\mathbf{v}$ are the same whether measured in S or S'. Thus

$$y' = y \quad \text{and} \quad z' = z \tag{1.33}$$

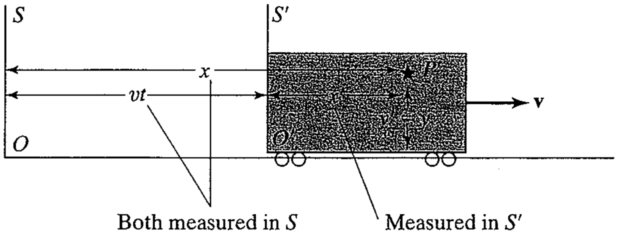


FIGURE 1.9

The coordinate x' is measured in S'. The distances x and vt are measured at the same time t in the frame S.

exactly as in the Galilean transformation. In finding x', it is important to keep careful track of the frames in which the various quantities are measured; in addition, it is helpful to arrange that the explosion whose coordinates we are discussing produces a small burn mark on the wall of the train at the point P' where it occurs. The horizontal distance from the origin O' to the mark at P', as measured in S', is precisely the desired coordinate x'. Meanwhile, the same distance, as measured in S, is $x - vt$ (since x and vt are the horizontal distances from O to P' and O to O' at the instant t, as measured in S). Thus according to the length-contraction formula (1.29),

$$x - vt = \frac{x'}{\gamma}$$

or

$$x' = \gamma(x - vt) \tag{1.34}$$

This gives x' in terms of x and t and is the third of our four required equations. Notice that if v is small, $\gamma \approx 1$ and the relation (1.34) reduces to the first of the Galilean relations (1.31), as required.

Finally, to find t' in terms of x, y, z, and t, we use a simple trick. We can repeat the argument leading to (1.34) but with the roles of S and S' reversed. That is, we let the explosion burn a mark at the point P on a wall fixed in S, and arguing as before, we find that

$$x = \gamma(x' + vt') \tag{1.35}$$

[This can be obtained directly from (1.34) by exchanging x, t with x', t' and replacing v by $-v$.] Equation (1.35) is not yet the desired result, but we can combine it with (1.34) to eliminate x' and find t'. Inserting (1.34) in (1.35), we get

$$x = \gamma[\gamma(x - vt) + vt']$$

Solving for t' we find that

$$t' = \gamma t - \frac{\gamma^2 - 1}{\gamma v} x$$

or, after some algebra (Problem 1.37),

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{1.36}$$

This is the required expression for t' in terms of x and t. When v/c is much smaller than 1, we can neglect the second term, and since $\gamma \approx 1$, we get $t' \approx t$, in agreement with the Galilean transformation, as required.

Collecting together (1.33), (1.34), and (1.36), we obtain our required four equations.

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}} \tag{1.37}$$

Hendrik Lorentz

(1853–1928, Dutch)

Lorentz was the first to write down the equations we now call the Lorentz transformation, although Einstein was the first to interpret them correctly. He also preceded Einstein with the length contraction formula (though, again, he did not interpret it correctly). He was one of the first to suggest that electrons are present in atoms, and his theory of electrons earned him the 1902 Nobel Prize in physics.

These equations are called the **Lorentz transformation**, or **Lorentz–Einstein transformation**, in honor of the Dutch physicist Lorentz, who first proposed them, and Einstein, who first interpreted them correctly. The Lorentz transformation is the correct relativistic modification of the Galilean transformation (1.31).

If one wants to know x, y, z, t in terms of x', y', z', t', one can simply exchange the primed and unprimed variables and replace v by $-v$, in the now familiar way, to give

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned} \tag{1.38}$$

These equations are sometimes called the **inverse Lorentz transformation.**

The Lorentz transformation expresses all the properties of space and time that follow from the postulates of relativity. From it, one can calculate all of the kinematic relations between measurements made in different inertial frames. In the next two sections we give some examples of such calculations.

1.12 Applications of the Lorentz Transformation

In this section we give three examples of problems that can easily be analyzed using the Lorentz transformation. In the first two we rederive two familiar results; in the third we analyze one of the many apparent paradoxes of relativity.

Example 1.4

Starting with the equations (1.37) of the Lorentz transformation, derive the length-contraction formula (1.29).

Notice that the length-contraction formula was used in our derivation of the Lorentz transformation. Thus this example will not give a new proof of length contraction; it will, rather, be a consistency check on the Lorentz transformation, to verify that it gives back the result from which it was derived.

Let us imagine, as before, measuring the length of a train (frame S') traveling at speed v relative to the ground (frame S). If the coordinates of the back and front of the train are x'_1 and x'_2, as measured in S', the train's proper length (its length as measured in its rest frame) is

$$l_0 = l' = x'_2 - x'_1 \tag{1.39}$$

To find the length l as measured in S, we carefully position two observers on the ground to observe the coordinates x_1 and x_2 of the back and front of the train at some convenient time t. (These two measurements must, of course, be made at the same time t.) In terms of these coordinates, the length l as measured in S is (Fig. 1.10)

$$l = x_2 - x_1$$

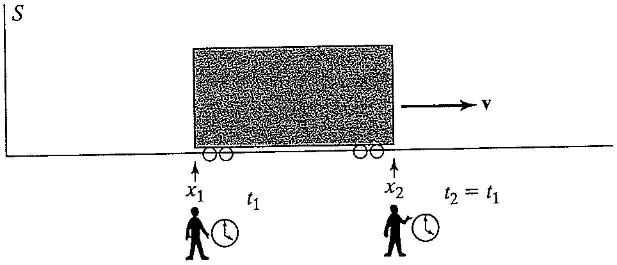


FIGURE 1.10

If the two observers measure x_1 and x_2 at the same time ($t_1 = t_2$), then $l = x_2 - x_1$.

Now, consider the following two events, with their coordinates as measured in S.

Event	Description	Coordinates in S
1	Back of train passes first observer	x_1, t_1
2	Front of train passes second observer	$x_2, t_2 = t_1$

We can use the Lorentz transformation to calculate the coordinates of each event as observed in S'.

Event	Coordinates in S'
1	$x'_1 = \gamma(x_1 - vt_1)$
2	$x'_2 = \gamma(x_2 - vt_2)$

(We have not listed the times t'_1 and t'_2 since they don't concern us here.) The difference of these coordinates is

$$x'_2 - x'_1 = \gamma(x_2 - x_1) \tag{1.40}$$

(Notice how the times t_1 and t_2 cancel out because they are equal.) Since the two differences in (1.40) are respectively $l' = l_0$ and l, we conclude that $l_0 = \gamma l$ or

$$l = \frac{l_0}{\gamma}$$

as required.

Example 1.5

Use the Lorentz transformation to rederive the time-dilation formula (1.18).

In our discussion of time dilation we considered two events, a flash and a beep, that occurred at the same place in frame S',

$$x'_{\text{flash}} = x'_{\text{beep}}$$

The proper time between the two events was the time as measured in S',

$$\Delta t_0 = \Delta t' = t'_{\text{beep}} - t'_{\text{flash}}$$

To relate this to the time

$$\Delta t = t_{\text{beep}} - t_{\text{flash}}$$

as measured in S, it is convenient to use the inverse Lorentz transformation (1.38), which gives

$$t_{\text{beep}} = \gamma\left(t'_{\text{beep}} + \frac{vx'_{\text{beep}}}{c^2}\right)$$

and

$$t_{\text{flash}} = \gamma\left(t'_{\text{flash}} + \frac{vx'_{\text{flash}}}{c^2}\right)$$

If we take the difference of these two equations, the coordinates x'_{beep} and x'_{flash} drop out (since they are equal) and we get the desired result,

$$\Delta t = t_{\text{beep}} - t_{\text{flash}} = \gamma(t'_{\text{beep}} - t'_{\text{flash}}) = \gamma\,\Delta t_0$$

Example 1.6

A relativistic snake of proper length 100 cm is moving at speed $v = 0.6c$ to the right across a table. A mischievous boy, wishing to tease the snake, holds two hatchets 100 cm apart and plans to bounce them simultaneously on the table so that the left hatchet lands immediately behind the snake's tail. The boy argues as follows: "The snake is moving with $\beta = 0.6$. Therefore, its length is contracted by a factor

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.36}} = \frac{5}{4}$$

and its length (as measured in my rest frame) is 80 cm. This implies that the right hatchet will fall 20 cm in front of the snake, and the snake will be unharmed." (The boy's view of the experiment is shown in Fig. 1.11.) On the other hand, the snake argues thus: "The hatchets are approaching me with $\beta = 0.6$, and the distance between them is contracted to 80 cm. Since I am 100 cm long, I will be cut in pieces when they fall." Use the Lorentz transformation to resolve this apparent paradox.

Let us choose two coordinate frames as follows: The snake is at rest in frame S' with its tail at the origin $x' = 0$ and its head at $x' = 100\,\text{cm}$. The two hatchets are at rest in frame S, the left one at the origin $x = 0$ and the right one at $x = 100\,\text{cm}$.

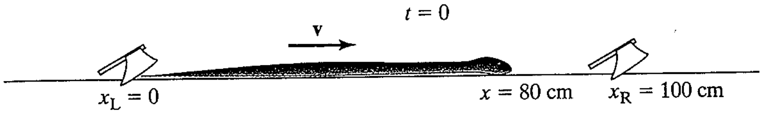


FIGURE 1.11

As seen in the boy's frame S, the two hatchets bounce simultaneously (at $t = 0$) 100 cm apart. Since the snake is 80 cm long, it escapes injury.

As observed in frame S, the two hatchets bounce simultaneously at $t = 0$. At this time the snake's tail is at $x = 0$ and his head must therefore be at $x = 80$ cm. [You can check this using the transformation $x' = \gamma(x - vt)$; with $x = 80$ cm and $t = 0$, you will find that $x' = 100$ cm, as required.] Thus, as observed in S, the experiment is as shown in Fig. 1.11. In particular, the boy's prediction is correct and the snake is unharmed. Therefore, the snake's argument must be wrong.

To understand what is wrong with the snake's argument, we must examine the coordinates, especially the times, at which the two hatchets bounce, as observed in the frame S'. The left hatchet falls at $t_L = 0$ and $x_L = 0$. According to the Lorentz transformation (1.37), the coordinates of this event, as seen in S', are

$$t'_L = \gamma\left(t_L - \frac{vx_L}{c^2}\right) = 0$$

and

$$x'_L = \gamma(x_L - vt_L) = 0$$

As expected, the left hatchet falls immediately beside the snake's tail, at time $t'_L = 0$, as shown in Fig. 1.12(a).

On the other hand, the right hatchet falls at $t_R = 0$ and $x_R = 100$ cm. Thus, as seen in S', it falls at a time given by the Lorentz transformation as

$$t'_R = \gamma\left(t_R - \frac{vx_R}{c^2}\right) = \frac{5}{4}\left(0 - \frac{(0.6c) \times (100\text{ cm})}{c^2}\right) = -2.5\text{ ns}$$

We see that, as measured in S', the two hatchets *do not fall simultaneously*. Since the right hatchet falls before the left one, it does not necessarily have to hit the snake, even though they were only 80 cm apart (in this frame). In fact, the position at which the right hatchet falls is given by the Lorentz transformation as

$$x'_R = \gamma(x_R - vt_R) = \tfrac{5}{4}(100\text{ cm} - 0) = 125\text{ cm}$$

and, indeed, the hatchet misses the snake, as shown in Fig. 1.12(b).

The resolution of this paradox and many similar paradoxes is seen to be that two events which are simultaneous as observed in one frame are not necessarily simultaneous when observed in a different frame. As soon as one recognizes that the two hatchets fall at different times in the snake's rest frame, there is no longer any difficulty understanding how they can both miss the snake.

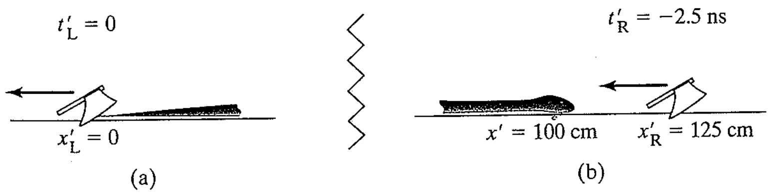


FIGURE 1.12

As observed in S', both hatchets are moving to the left. The right hatchet falls before the left one, and even though the hatchets are only 80 cm apart, this lets them fall at positions that are 125 cm apart.