Conservation of momentum

Two identical particles A and B head-on collide in their CM frame and stick together at the origin. However, they each moves not along the x-axis, but a little tilted from the x-axis, on the xy plane.

Then we study the case from the frame A' which moves along $+x$ axis *w.r.t.* the CM frame so that the particle only moves upward from the negative y axis at $\boldsymbol{u}_A = \ell\, \boldsymbol{i}$ without x movement. The B particle has both x and y movements.

$$\boldsymbol{u}_B = -w\boldsymbol{j} - v\boldsymbol{i}\ , \quad \boldsymbol{u}_A = +\ell\ \boldsymbol{j}\ ,\ \text{in the } A' \text{ frame.}$$

Similar formulas can be obtained by symmetry in the B' frame where the B particle only moves downward from the positive y axis without x movement.

$$\boldsymbol{u}_A = +w\boldsymbol{j} + v\boldsymbol{i}\ , \quad \boldsymbol{u}_B = -\ell\ \boldsymbol{j}\ ,\ \text{in the } B' \text{ frame.}$$

As the frame B' moves at $-v\boldsymbol{i}$ *w.r.t.* A', we relate the y velocity components of the B particle, $w = \ell\sqrt{1 - v^2/c^2}$. The B particle speed u in the A' frame is

$$u^2 = v^2 + w^2 = v^2 + \ell^2(1 - v^2/c^2)\ , \quad 1 - u^2/c^2 = (1 - \ell^2/c^2)(1 - v^2/c^2)\ .$$

The Newtonian ideal of a mass independent of the velocity fails to respect the conservation of momentum. Let us study how the mass depending on its velocity preserves the principle,

$$m_\ell \ell = m_u w\ , \quad m_\ell \ell = m_u \ell\sqrt{1 - v^2/c}\ , \quad m_\ell = m_u\sqrt{1 - v^2/c}\ .$$

$$m_\ell\sqrt{1 - \ell^2/c^2} = m_u\sqrt{1 - u^2/c} \equiv m$$

The first equality demands the products equal to a common constant, which is names m, the rest mass.

$$\boxed{m_u = m/\sqrt{1 - u^2/c^2}}$$

Let us now look at the composite AB which moves $\boldsymbol{u}_{AB} = +r\boldsymbol{i}$ in the B' frame, but $-r\boldsymbol{i}$ in the A' frame. As A' moves $v\boldsymbol{i}$ in B', the velocity transforms as

$$-r = \frac{r - v}{1 - rv/c^2}\ , \quad (v/c^2)r^2 - 2r + v = 0\ .$$

$$r = \frac{1 - \sqrt{1 - v^2/c^2}}{v/c^2}\ ,\ \text{with the other root nonsense. So}\ \ r = \frac{v}{1 + \sqrt{1 - v^2/c^2}}\ .$$

$$r = \frac{m_u v}{m_u + m_u\sqrt{1 - v^2/c^2}} = \frac{m_u v}{m_u + m_\ell}$$

Conservation of x momentum gives $(M_{AB})_r r = m_u v$. This is only true if

$$(M_{AB})_r = m_u + m_\ell\ .$$

Defining generically the relativistic energy E as velocity dependent mass times c^2, *i.e.*

$$E_u = m_u c^2 = \frac{mc^2}{\sqrt{1 - u^2/c^2}}\ ,\ E_\ell = m_\ell c^2 = \frac{mc^2}{\sqrt{1 - \ell^2/c^2}}\ ,\ E_{AB} = (M_{AB})_r c^2 = \frac{(M_{AB})c^2}{\sqrt{1 - u^2/c^2}}\ .$$

We have the conservation of relativistic energy $\boxed{E_{AB} = E_A + E_B}$.