Time dilation without using a photon clock

Wai-Yee Keung
Physics Department, University of Illinois at Chicago, Illinois 60607

In the usual textbook explanation[1, 2, 3] of the time dilation in special relativity, a clock device made of a bouncing photon is adopted to measure time intervals. The dilation factor is obtained by placing the bouncing direction vertical to the relative motion between frames. The derivation is at the introductory level, but it requires the use of Pythagorean theorem in planar geometry. In this article, we show another simple (or simpler) approach which restricts our attention to only one dimensional space plus time.

Our illustration reinforce the concept of the proper time. The time interval between two events at the same spatial coordinates is call proper, otherwise non-proper. We understand that these two time measurements of the same two events by a proper observer and a non-proper observer are related proportionally with a coefficient γ that depends only on their relative speed.

$$t(\text{non-proper}) = \gamma t(\text{proper})$$

Let the observer O measure the time by his rest clock which ticks. The first event READY occurs at O who initiates his clock $t = 0$ when another observer O' just passes by and moves to the right at the speed v. "SENT" denotes the second event that a light signal is sent to the moving observer O' from O at Δt of his clock. The third event RECEIVE happens at O' when he receives the signal.

We use two methods for calculating the overall time between READY and RECEIVE.

The first method is straightforward, completely in terms of what O measures. The overall time measured by O is

$$\Delta t + \frac{\Delta t v}{c - v} \ .$$

The denominator $c - v$ is based on arithmetics in the grade school on the problem about the rabbit catching up the turtle.

Now, we adapt the viewpoint of O', who claims that the overall time is also the sum of READY-SENT $\gamma\Delta t$ and SENT-RECEIVE $v\gamma\Delta t/c$,

$$\gamma\Delta t + v\gamma\Delta t/c \ .$$

Note that Δt is the proper time of O which becomes a non-proper time interval $\gamma\Delta t$ measured by O' system. The proportional factor γ is to be determined by consistency of relativity.

As the time interval between READY and RECEIVE is proper to O', we can use the principle of relativity to claim the relation,

$$\Delta t + \frac{\Delta t v}{c - v} = \gamma(\gamma\Delta t + v\gamma\Delta t/c) \ .$$

We find

$$\frac{c}{c - v} = \gamma^2(1 + v/c) \ , \qquad \gamma = 1/\sqrt{1 - v^2/c^2} \ .$$

The following table clarifies the nature of proper and non-proper times.

Event pair	measured by O	measured by O'
READY-SENT	Δt proper	$\gamma\Delta t$ non-proper
SENT-RECEIVE	$\Delta tv/(c-v)$ non-proper	$\gamma\Delta tv/c$ non-proper
READY-RECEIVE	$\Delta t + \Delta tv/(c-v)$ non-proper	$\gamma\Delta t + \gamma\Delta tv/c$ proper

Simultaneity

Let SET be the event synchronized with SENT at $t = \Delta t$ from the viewpoint of O, but positioned at $x = v\Delta t$ where O' is instantaneously located. The event pair READY–SET are proper according to O', but non-proper according to O.

$$t(\text{SET}) = \Delta t \ , \quad t'(\text{SET}) = \Delta t/\gamma \ .$$

We have

$$t'(\text{SENT}) - t'(\text{SET}) = \gamma\Delta t - \Delta t/\gamma = \gamma\Delta t v^2/c^2 = (xv/c^2)/\sqrt{1-v^2/c^2} \ .$$

Therefore two simultaneous events at a distance x apart observed by O happen to have time separation

$$t'(\text{asynchrony}) = (vx/c^2)/\sqrt{1-v^2/c^2} \ ,$$

as observed by another observer O' who moves at a relative v with respect to O, along the x direction.

Length Contraction

The trajectory of a point object in space-time constitutes a world line. A point object at rest also registers a world line because its time coordinate changes. Two world lines are parallel if there is a common rest frame for them. The length contraction is a property about a pair of parallel world lines. Their spatial separation (ℓ_0) defined in their common rest frame is called the proper length. An observer at velocity v with respect to the rest frame of this pair of parallel world lines measures their separation ℓ in a *simultaneous snapshot*.

We reuse the previous example of READY and SENT. Let these events accompany flashes which leave permanent marks on the ruler in the frame of O'. The mark associated with READY is located at the origin of O' and the mark with SENT is located at a distance $\ell_0 = v\gamma\Delta t$. Note that these two marks consititute two stationary world lines as observed by O'. In the frame of O, these two marks are moving, separated by a distance $\ell = v\Delta t$. Therefore we find the length contraction relation

$$\ell = \ell_0\sqrt{1-v^2/c^2} \ .$$

In conclusion, we have rederived the three fundamental relations of the time dilation, the length contraction, and simultaneity in special relativity based on the concept of proper and non-proper time intervals. We have avoided using the photon clock oriented in the transverse direction which is not essential because the relativistic transformation exists even in one-dimensional system as expected.

References

[1] W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley, Reading, MA 1962), 2nd Ed., pp 289-290.

[2] R.B. Leighton *Principle of Modern Physics* (McGraw-Hill, New York, 1959) pp 12-13.

[3] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics* I, pp. 15-4, Addison Wesley (1968).