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Competing market makers, liquidity provision, and bid–ask spreads☆

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Abstract

This paper develops a dynamic market microstructure model of liquidity provision in which M strategic market makers compete in price schedules for order flow from informed and uninformed traders. In equilibrium, market makers post price schedules that are steeper than efficient ones, and the market bid–ask spreads can be decomposed into two components, one due to adverse selection and the other due to imperfect competition. At any time, the two components are proportional to each other with a coefficient of proportionality depending on M. Several testable hypothesis are derived regarding the time-series and cross-sectional properties of prices and the bid–ask spreads. In particular, a new empirical measure of market competitiveness is proposed which can be estimated from the history of transaction prices and trading volumes. Finally, the properties of continuous market are also investigated. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

In securities markets liquidity plays a fundamental role because it facilitates efficient risk sharing and encourages the collection of costly information. Many financial markets rely on dealers, or market makers, for the provision of liquidity. The trading behavior of market makers affects the short-term dynamics of securities prices and, therefore, it is of great interest to traders, regulators, and researches.

Earlier market microstructure papers have often considered market makers' trading behavior as perfectly competitive. The classic models of asymmetric information (see Kyle, 1985; Glosten and Milgrom, 1985; Easley and O'Hara, 1987) focus on the role of adverse selection created by the presence of better informed traders on price formation, but they do not deal with the strategic aspects of market makers' behavior. In these models, market makers are simply assumed to be perfect competitors who provide liquidity at prices that earn them a zero profit. The zero-profit assumption is a convenient abstraction which greatly simplifies the game–theoretical analysis of the models, but it is often at odds with the empirical facts about securities markets. These facts indicate that market makers often post rather noncompetitive prices and that they do earn positive profits.[1] Furthermore, the standard competitive models cannot explain how market makers may be able to cover substantial fixed costs associated with making a market in a security.

Recently, there has been a rapidly growing literature to investigate the strategic trading behavior of market makers in securities markets under adverse selection. Glosten (1989) studies properties of a monopolistic specialist system as opposed to a competitive specialist system. Kavajecz (1998) extends Glosten by allowing the specialist to choose quantities of trades as well as their prices. Glosten (1994) examines the limiting case where the number of liquidity providers approaches infinity. Dennert (1993), Bernhardt and Hughson (1997), and Biais et al. (2000) analyze price competition among market makers, when informed and uninformed traders are allowed to split their orders between markets.[2] All of the above, however, analyze market makers' trading behavior in a *static* framework, at a point in time, and do not address the effects of such a behavior on transaction prices over time.

In contrast, the present paper studies trading behavior of market makers in a *dynamic* microstructure model with asymmetric information. In this model we extend the Kyle (1985) sequential market to allow imperfect competition of M

[1] For example, Christie and Schultz (1994) and Christie et al. (1994) find empirical evidence of noncompetitive behavior of the Nasdaq dealers. Hasbrouck and Sofianos (1993) and Sofianos (1995) report on profitability of the NYSE specialists.

[2] See Seppi (1997), Bondarenko and Sung (2000) for single auction models of a strategic specialist facing competition from the limit order book.

profit maximizing market makers. Market makers compete in price schedules and provide liquidity for two types of traders: an informed trader who knows an asset liquidation value in advance and a number of liquidity traders.[3]

One can view our modeling of competing market makers in several ways. First, it can be interpreted as competition between dealers on the same exchange such as competition of the Nasdaq dealers or specialists ("jobbers") on the trading floor of the London Stock Exchange. Other important examples of multiple dealer markets include the foreign exchange market and the secondary market for U.S. Treasury bills. Second, it can be interpreted as intermarket competition for order flow between different exchanges. For example, some stocks trade concurrently on the NYSE, the regional exchanges, the over-the-counter market, or a growing number of electronic communication networks (ECNs). Third, even on the trading floor of the NYSE (where a single specialist is assigned to make market in each stock), specialists are not total monopolists because they face competition from floor traders and limit order submitters. Therefore, our model may also provide some insights into the nature of that competition.[4]

One of the objectives of this paper is to investigate whether findings of single period models of imperfect competition are robust to the introduction of multiple rounds of trading. This is important because, as stressed in Black (1991), characterization of equilibrium in microstructure models may be flawed if dynamic issues are ignored.[5] Another objective is to study the consequences of imperfect competition of market makers for the intertemporal properties of transaction prices and to derive testable restrictions. The main theoretical results are the following:

1. We show that when the number of market makers equals or exceeds three, there exists a unique linear equilibrium. The equilibrium is symmetric and characterized by simple recursive strategies of the market participants. Market makers post price schedules that are steeper than the efficient ones and earn strictly positive expected profits. In equilibrium, each market maker behaves as a monopolist facing a residual demand curve resulting from the maximizing behavior of the informed trader and price schedules offered by the competitors. As competition intensifies, the market becomes deeper, and trading costs of

[3] In Kyle's model, there are three types of risk-neutral agents: an informed trader, liquidity traders and a market maker. Later, his model has been extended to accommodate richer market structures. Foster and Viswanathan (1993) and Holden and Subrahmanyam (1992) develop models with multiple informed traders. Back (1992) studies a continuous-time generalization. Subrahmanyam (1991) extends the basic model to allow for a risk-averse insider. Strategic behavior of liquidity traders is studied in Admati and Pfleiderer (1988).

Single-period models where liquidity suppliers compete in demand schedules are introduced in Kyle (1984) and Kyle (1989).

[4] See O'Hara and Oldfield (1986) for a discussion of different market organizations.

[5] See also the discussion on dynamic issues and intuitive examples in Glosten (1994).

liquidity traders becomes smaller. This is because increased competition diminishes the ability of each market maker to extract rents by reducing the supply of liquidity. We also demonstrate that, in the limiting case when M approaches infinity, market makers' profits go to zero and the equilibrium converges to the sequential auction equilibrium of Kyle (1985).

2. We argue that the presence of the informed trader in the market is critical for the market viability. Intuitively, in a market without asymmetric information, competition among market makers drives their expected profits to zero. In this case, the residual demand curve faced by each market maker is infinitely elastic and the Bertrand outcome arises. This means that, if there are fixed costs associated with making a market, then no one will be willing to provide liquidity and the market will cease to function. In contrast, when the informed trader is present, the competition among market makers becomes limited, as long as their number is finite and market makers post price schedules that earn them strictly positive expected profits. In fact, the expected profit of each market maker is the higher the more severe the adverse selection problem, implying that market makers actually *prefer* more information asymmetry to less. Furthermore, when there are fixed costs associated with market making, some minimal amount of adverse selection is absolutely necessary in order to keep the market open.

3. Several new comparative statics results are derived for equilibrium outcomes. Some of these results refine properties of important Kyle (1985) and related models. For example, we prove that the informed trading intensity, market depth, and price variance change monotonically as the announcement date approaches provided that the level of liquidity trading is constant over time. However, these statistics may become nonmonotonic if the level of liquidity trading is time-varying.

4. When the intervals between auctions go to zero, the sequential auction approaches continuous trading. In this case, the participants' equilibrium strategies are obtained in closed form. We show that the continuous market with a finite number of market makers is infinitely tight and, compared with the continuous market in Kyle (1985), it has the same resiliency but lower market depth.

Our model also has a number of empirical implications, regarding both the time-series and cross-sectional properties of transaction prices and the bid–ask spreads. In particular, the bid–ask spreads in our market can be decomposed into two components, one due to adverse selection in the market and the other due to imperfect competition between market makers. Because of the adverse selection component, asset prices gradually incorporate the informed trader's private information and eventually converge to the asset's true liquidation value. The presence of the imperfect competition component induces negative serial correlation in transaction prices. At any time, the two components are proportional to each other with a coefficient of proportionality depending on M, implying that the number of market makers is an important cross-sectional determinant of the bid–ask spread.

We use the decomposition of the bid–ask spread to construct a new measure of market competitiveness. This empirically relevant measure can be estimated from the history of transaction prices and volumes. It can be interpreted as the *effective* number of competing liquidity providers consistent with observed price dynamics. For example, when the effective number of liquidity providers estimated from the transactions data is high, a market can be viewed as very competitive, and vice versa. The construction of the new measure is based on the idea that change in transaction prices is positively correlated to the current order size but negatively correlated to the preceding order size and that the two correlation coefficients are linked to each other through the number of market makers.

We argue that the new measure may be used to empirically distinguish between *noncooperative* competition and *explicit* collusion of market makers. Intuitively, our analysis shows that competing market makers, acting in their own self-interest, can sustain the bid–ask spreads above competitive levels. Still, the bid–ask spreads resulting from noncooperative but imperfect competition will usually be much narrower than those resulting from explicit collusion, in which market makers cooperate to fix prices. To put it differently, our model shows what "normal" profits of noncolluding market makers should be. If the actual profits of market makers (as well as market spreads) are found to be much higher than those predicted by the theory, one could interpret these findings as an evidence of a possible collusion among market makers.

In this respect, our model has important policy implications and may shed some light on the recent controversy over the spreads in the Nasdaq market. In two widely publicized papers, Christie and Schultz (1994) and Christie et al. (1994) conclude that Nasdaq dealers tacitly cooperated to fix the bid–ask spread. The first paper documents an absence of odd-eighth quotes for Nasdaq stocks while the second one reports a significant drop in spreads following the public disclosure of the results of the first paper. These findings have led to investigations of Nasdaq dealer pricing by the Justice Department and the Security and Exchange Commission. It is important to note that the two aforementioned studies, while demonstrating peculiar pricing patterns in the Nasdaq market, do not prove that the spreads were, in some sense, too wide and that the dealers' profits were, in some sense, too high. However, we argue that if the dealers do collude, it is possible to detect the collusion by showing that the effective number of liquidity providers implicit in the dynamics of transaction prices is much smaller than the actual number of active dealers in the market.

The rest of the paper is organized as follows. The next section presents a sequential auction model and characterizes its unique linear equilibrium.

Sections 3.1–3.5 discuss properties of the equilibrium. Section 3.1 looks at the limiting case when the number of market makers M approaches infinity (perfect competition). Section 3.2 derives comparative statics for equilibrium outcomes. Section 3.3 suggests the decomposition of the bid–ask spread into components due to adverse selection and due to imperfect competition. It also derives several theoretical results regarding the time-series and cross-sectional properties of transactions prices and the bid–ask spreads. Empirical implications are discussed in Section 3.4. Section 3.5 investigates properties of continuous market when intervals between auctions approach zero. Sections 4 and 5 discuss and summarize the paper. The appendix contains proofs of our results.

2. Model

2.1. Notation and structure

The financial market model is a modification of the multi-period auction in Kyle (1985) to allow for multiple competing market makers.[6] Consider a market where a single risky asset is traded. The asset's current value is v_0. Trading takes place over a time interval which begins at $t = 0$ and ends at $t = 1$. There are N sequential auctions ($n = 1, \ldots, N$) which take place at discrete points of time $t_1, \ldots, t_N$ such that

$$0 = t_0 < t_1 < \cdots < t_N = 1.$$

After time 1, there will be a public release of information about the value of the asset. The asset's final value is denoted v^*, a normal random variable with mean v_0 and variance σ_v^2.

The market for the asset consists of three types of risk-neutral participants: (1) one large (institutional) informed trader who knows the asset liquidation value v^* in advance, (2) a number of liquidity traders who trade for purposes exogenous to the model, and (3) several market makers indexed $m = 1, \ldots, M$.

The informed and liquidity traders can only trade the asset via market makers. It is assumed that traders can costlessly and simultaneously send market orders to each market maker. Let Δx_{mn} and Δz_{mn} denote the orders by the informed trader and the aggregate order by all liquidity traders to the mth market maker at the nth auction, respectively. The total orders by the informed and liquidity traders at the nth auction are denoted $\Delta x_n = \sum_{i=1}^{M} \Delta x_{mn}$ and $\Delta z_n = \sum_{m=1}^{M} \Delta z_{mn}$, respectively. It is assumed that Δz_n is serially uncorrelated

[6] Alternatively, the model may be viewed as an extension of the static model in Bernhardt and Hughson (1997) to allow for multiple rounds of trading.

and normally distributed with zero mean and variance $\sigma_z^2 \Delta t_n$, where $\Delta t_n := t_n - t_{n-1}$.

At the beginning of the nth auction, market makers simultaneously set price schedules $P_{mn}(\cdot)$. Schedule $P_{mn}(\cdot)$ details the prices at which market maker-m is willing to handle any given order size. Let $p_{mn} = P_{mn}(\Delta x_{mn} + \Delta z_{mn})$ denote the clearing price set by market maker-m. Before the start of the nth auction, public information includes the history of past clearing prices of all market makers:

$$\mathscr{F}_{n-1} := \{p_{mj} \,|\, m = 1, \ldots, M; j = 1, \ldots, n-1\}.$$ [7]

The extensive form of the market game is as follows:

I. At time $t_0 = 0$, the informed trader learns the liquidation value v^*.

II. At time t_n, the nth auction takes place:
   1. M market makers simultaneously post trading schedules $P_{mn}(\cdot)$;
   2. The informed and liquidity traders simultaneously choose vectors of their trades $\{\Delta x_{mn}\}_{m=1}^M$ and $\{\Delta z_{mn}\}_{m=1}^M$ with M market makers;
   3. Realized prices p_{mn} are announced and trades are executed.

III. After $t = 1$, the liquidation value v^* is announced to the public and payoffs of all participants are realized.

For this market game we consider the Perfect Bayesian equilibria. These equilibria are characterized by the conditions that (i) all agents choose their strategies to maximize expected profits conditional on information they have and their conjectures about optimal strategies of other agents and (ii) conjectures of all agents are correct. More specifically, at the nth auction, when the informed and liquidity traders face price schedules $P_{mn}(\cdot)$, they must decide how much they want to trade with each market maker. Liquidity traders optimally split their order Δz_n between market makers to minimize their expected losses. The informed trader, when choosing his trades Δx_{mn}, takes into account the effect his demand has on prices in both the current auction and future auctions. To maximize the rent of his private information regarding v^*, the informed trader restricts his trading in earlier auctions to reap larger profits in future auctions. Finally, price schedules $P_{mn}(\cdot)$ are determined through (imperfect) competition of market makers.

In this paper, we focus on the popular class of linear market equilibria in which market makers' price schedules and strategies of the informed and liquidity traders are linear functions of their information. Specifically, in linear

[7] As will be seen later, in equilibrium, clearing prices of all market makers are the same, i.e., $p_{1n} = \cdots = p_{Mn} = p_n$.

equilibria, participant's strategies at the nth auction are

$$\Delta x_{mn} = \beta_{mn} v^* \Delta t_n + \alpha_{mn},$$

$$p_{mn} = \lambda_{mn}(\Delta x_{mn} + \Delta z_{mn}) + \omega_{mn},$$

$$\Delta z_{mn} = \gamma_{mn} \Delta z_n + \delta_{mn} \quad \text{such that} \quad \sum_{m=1}^{M} \Delta z_{mn} = \Delta z_n,$$

where β_{mn}, α_{mn}, λ_{mn}, ω_{mn}, γ_{mn}, and δ_{mn} are constants (which may depend on the information available to participants at time $t-1$).

In the next section, we show that, when $M > 2$, there exists a unique linear equilibrium. In the equilibrium, market makers post price schedules that are steeper than efficient ones and earn strictly positive expected profits. As long as the number of market makers is finite, their competition is not perfect, and the Bertrand outcome does not arise. To see intuitively why competition is limited in our model, suppose to the contrary that all market makers post competitive price schedules that earn them zero expected profits. To reflect the informational content of trades in the presence of the better-informed trader, these price schedules must be upward-sloping. Then market maker-1 can ensure positive profits by offering a slightly steeper price schedule. By doing so, he does not lose his *entire* order to other market makers, but only a small portion of it. This is because the informed and liquidity traders, who face upward-sloping price schedules, split their orders in such a way that marginal trading costs are the same across all market makers.

More importantly, in response to a steeper price schedule, the informed trader reduces his orders to of market maker-1 by more than liquidity traders do (in other words, the demand of the profit-motivated informed trader is more elastic than that of liquidity traders). As a result, market maker-1 will earn strictly positive profits, and the Bertrand outcome cannot arise. Interestingly enough, it is the presence of the informed trader that destroys the perfect competition among market makers. We will return to this issue in the following sections.

2.2. Equilibrium

To characterize market equilibria we use the following notation.

(1) Conditional expected value and variance of the asset liquidation value after the nth auction:

$$v_n := E[v^* \mid \mathscr{F}_n], \quad \sigma_n^2 := \mathrm{var}\{v^* \mid \mathscr{F}_n\}.$$

In this notation $\sigma_0 = \sigma_v$. Also, let $\rho_n := \sigma_z / \sigma_n$ denote the ratio of liquidity trading to the asset value uncertainty after the nth auction. Intuitively, "noise-to-signal" ratio ρ_n measures the quality of information in the market and is inversely related to the degree of adverse selection. For example, securities with high asset value uncertainty and low level of liquidity trading have small ρ (implying severe adverse selection).

(2) *Aggregate* sensitivity of the informed trader's strategy β_n and the slope of the *effective* price schedule provided by all M market makers λ_n:

$$\beta_n := \sum_{m=1}^{M} \beta_{mn}, \quad \frac{1}{\lambda_n} := \sum_{m=1}^{M} \frac{1}{\lambda_{mn}}, \tag{1}$$

(3) Profits of the informed trader and market maker-m in auctions $n, \ldots, N$:

$$\pi_n^I := \sum_{j=n}^{N} \sum_{m=1}^{M} (v^* - p_{mj})\Delta x_{mj}, \quad \pi_{mn}^M := \sum_{j=n}^{N} (p_{mj} - v^*)(\Delta x_{mj} + \Delta z_{mj}),$$

Now we can state the main result of this section.

Proposition 1. Let $M > 2$. Then there exists a unique linear equilibrium. The equilibrium is symmetric and given by the following recursive structure. For all auctions $n = 1, \ldots, N$ and for all market makers $m = 1, \ldots, M$ there exist constants β_n, λ_n, a_n, b_n, and c_n such that

$$\Delta z_{mn} = \frac{1}{M}\Delta z_n, \tag{2}$$

$$\Delta x_{mn} = \frac{\beta_n}{M}(v^* - v_{n-1})\Delta t_n, \tag{3}$$

$$p_{mn} = v_{n-1} + \lambda_n M(\Delta x_{mn} + \Delta z_{mn}), \tag{4}$$

$$E[\pi_n^I \mid \mathscr{F}_{n-1}, v^*] = a_{n-1}(v^* - v_{n-1})^2 + b_{n-1}, \tag{5}$$

$$E[\pi_{mn}^M \mid \mathscr{F}_{n-1}] = c_{n-1}. \tag{6}$$

Given $\rho_0 = \sigma_z/\sigma_v$, constants β_n, λ_n, ρ_n, a_n, b_n, and c_n are uniquely determined by the following difference equation system:

$$\lambda_n = \frac{M-1}{M-2}\frac{\beta_n}{\rho_n^2}, \tag{7}$$

$$\beta_n \Delta t_n = \frac{1 - 2a_n\,\beta_n/\rho_n^2}{2\lambda_n - 2a_n\,\beta_n^2/\rho_n^4}, \tag{8}$$

$$a_{n-1} = a_n\left(1 - \frac{\beta_n^2}{\rho_n^2}\Delta t_n\right)^2 + (1 - \lambda_n\beta_n\Delta t_n)\beta_n\Delta t_n, \tag{9}$$

$$b_{n-1} = b_n + a_n \frac{\beta_n^2}{\rho_n^4} \sigma_z^2 \Delta t_n, \tag{10}$$

$$c_{n-1} = c_n + \frac{\sigma_z^2 \Delta t_n}{M(M-2)} \frac{\beta_n}{\rho_{n-1}^2}, \tag{11}$$

$$\rho_{n-1}^2 = \rho_n^2 - \beta_n^2 \Delta t_n, \tag{12}$$

subject to $a_N = b_N = c_N = 0$ *and the second order condition*

$$\lambda_n - a_n \frac{\beta_n^2}{\rho_n^4} > 0. \tag{13}$$

When $M \leqslant 2$, *there is no linear equilibrium.*

The proof of Proposition 1 is a modification of the original proof in Kyle (1985). There is a key difference between the two proofs, however. In Kyle (1985), a single market maker and liquidity traders are "passive" agents. The market maker's strategy is predetermined by the market efficiency (or zero profit) condition; liquidity traders' decision does not arise at all. Therefore, only strategic decisions of the informed trader need to be considered in this model.

In contrast, in our model market makers are not restricted to zero-profit price schedules. In fact, market maker-m chooses optimal price schedule to maximize his profit given strategies of the informed and liquidity traders as well as price schedules of other market makers. In doing so, market maker-m must, in particular, know how the informed and liquidity traders optimally split their orders when they face different price schedules. This introduces a new level of complexity to the game–theoretical analysis because the resulting equilibrium must incorporate strategic interactions of all market participants.[8]

An interesting result in Proposition 1 is that *no* linear equilibrium exists when $M \leqslant 2$. The nonexistence result is familiar from the static model of Dennert (1993). It turns out that when there are only two market makers, each

[8] The proof of Proposition 1 is slightly more general than the corresponding proofs in Dennert (1993) and Bernhardt and Hughson (1997) for their single auction models. In particular, Dennert (1993) uses the simplifying assumption that in each moment only one trader (who could be either informed, with probability μ, or uninformed, with probability $1-\mu$) trades with market makers.

Bernhardt and Hughson (1997), to facilitate their analysis, assume that market makers' price schedules $P_m(\cdot)$ and the informed trader's strategy Δx_m are such that $P_m(0) = E[v^* \mid \mathscr{F}_0]$, and $\Delta x_m = \beta_m(v^* - E[v^* \mid \mathscr{F}_0])$. This assumption is relatively innocuous in a single auction model but it is more restrictive in a multiple auction setting because it stipulates that, at any period, the participants' strategies can only depend on the asset's expected value at that time. This, for example, excludes strategies that are functions of the asset's past prices.

of them always has an incentive to post a steeper price schedule than the other one so that no equilibrium can be sustained.

To see more clearly why an equilibrium exists for $M \geqslant 3$ but not for $M = 2$ (obviously, no equilibrium can exist with a monopolistic market maker), one needs to consider how the market participants respond when one of the market makers makes an out-equilibrium move. Suppose that, at the nth auction, market makers 2 to M set price schedules with $\lambda_{mn} = l$ and market maker-1 deviates by choosing $\lambda_{1n} = l' > l$. From the proof of Proposition 1, this deviation has the following four effects: (1) the slope of the effective price schedule of all market makers λ_n becomes steeper; (2) liquidity traders' demand in market-1 ($\Delta z_{1n} = \lambda_n / \lambda_{1n} \Delta z_n$) decreases; (3) the informed trader's aggregate demand (i.e., β_n) as well as his demand in market-1 ($\beta_{1n} = \lambda_n / \lambda_{1n} \beta_n$) both decrease; and (4) adverse selection in the future auctions (if $n < N$) becomes more severe (because the informed trader reveals less information at the nth auction). Market maker-1 benefits from effects (1), (3) and (4), but loses from effect (2). (In Section 4 we discuss in more detail exactly why market makers prefer more information asymmetry to less.)

Market maker-1' expected profit from the remaining auctions $E[\pi_{1n}^M \mid \mathscr{F}_{n-1}]$ can be decomposed into the sum of the expected gain from the liquidity traders at the nth auction, the expected loss to the informed trader at the nth auction, and the net expected profit *after* the nth auction $E[\pi_{1(n+1)}^M \mid \mathscr{F}_{n-1}]$. The last two components increase when market maker-1 chooses a steeper λ_{1n} (due to effects (1), (3) and (4)). The first component is $\lambda_n^2 / \lambda_{1n} \sigma_z^2 \Delta t_n$. Its partial derivative with respect to λ_{1n} is negative when $M \geqslant 3$, zero when $M = 2$, and positive when $M = 1$ because:

$$\frac{\partial}{\partial \lambda_{1n}} \left(\frac{\lambda_n^2}{\lambda_{1n}} \right) \Bigg|_{\lambda_{1n} = \lambda_{2n} = \cdots = l} = \frac{l}{M^3} (2 - M).$$

Therefore, for $M = 1, 2$ market maker-1 *always* has an incentive to increase λ_{1n} and thus no equilibrium can exist. In contrast, when $M \geqslant 3$, it is possible to find an l that exactly counterbalances the potential gains from the last two components and losses from the first component (i.e., when the derivative of $E[\pi_{1n}^M \mid \mathscr{F}_{n-1}]$ with respect to λ_{1n} is zero) so that market maker-1 has no incentive to deviate.

It may also be instructive to compare the equilibrium in Proposition 1 with the equilibrium of a simpler market game where *no* informed trader is present. In this game, M market makers compete for inelastic demands of liquidity traders and only effects (1) and (2) need to be considered. It is straightforward to verify that, when $M \geqslant 3$, there is a unique sequential linear equilibrium which describes the Bertrand outcome with zero slopes of price schedules $\lambda_{mn} = 0$ for all m and n. When $M = 2$, the game has infinitely many equilibria such that $\lambda_{1n} = \lambda_{2n} = l_n$ for any positive l_n. In the latter case, neither market maker has an incentive to deviate unilaterally, although they do benefit by *simultaneously* increasing the slope of their price schedules l_n. Finally, when $M = 1$ no equilibrium exists.

3. Properties of the equilibrium

In this section we examine properties of the linear equilibrium and illustrate its outcomes with a series of numerical examples. In all numerical examples, we assume that $\sigma_v = 1$, $\sigma_z = 1$, and that $\Delta t_n = 1/N$. Recall that β_n is the aggregate sensitivity of the informed trader's strategy and λ_n is the sensitivity of market makers' aggregate price schedule defined in (1). As in Kyle (1985), the asset's conditional variance σ_n^2 measures how much of the informed trader's private information is not yet incorporated in prices.

To get an initial insight into equilibrium outcomes, Fig. 1 plots λ_n, β_n, and σ_n^2 for the particular cases of $N = 4$, 20, 100 and ∞, when the number of market makers is $M = 3$. As can be seen from this figure, λ_n and σ_n^2 monotonically decline through time, while β_n rapidly increases with n. The next subsections look at the properties of the equilibrium in more detail.

3.1. Perfect competition, $M = \infty$

The following proposition states an intuitive result that the zero-profit equilibrium in the Kyle's (1985) sequential auction is the limiting case of our model when the number of market makers approaches infinity.

Proposition 2 (Perfect competition). *In the limiting case where M approaches infinity*, *the equilibrium in Proposition* 1 *converges to the zero-profit equilibrium in the sequential auction of Kyle* (1985).

Fig. 2 illustrates Proposition 2. It plots λ_n, β_n, and σ_n^2 for the particular cases of $M = 3,\ 5,\ 20$ and ∞, when the number of auctions is fixed at $N = 20$. As competition among market makers intensifies (M increases), they post less steep price schedules (λ_n decreases), the informed trader trades more aggressively (β_n increases), and private information is incorporated in prices more rapidly (σ_n^2 decreases).

Intuitively, as competition intensifies, each market maker handles smaller order flow, and the residual demand faced by each market maker becomes more elastic. Therefore, the expected profits of each individual market maker as well as of the whole group decrease. At the limit, the expected profit of all market makers is zero. In contrast, the welfares of the informed and liquidity traders improve. The expected profit of the informed trader $E[\pi^I]$, the expected profit of all market makers $E[\pi^M]$, and the expected loss of liquidity traders $E[\pi^L]$ are plotted in Fig. 3 as functions of the number of market makers M for

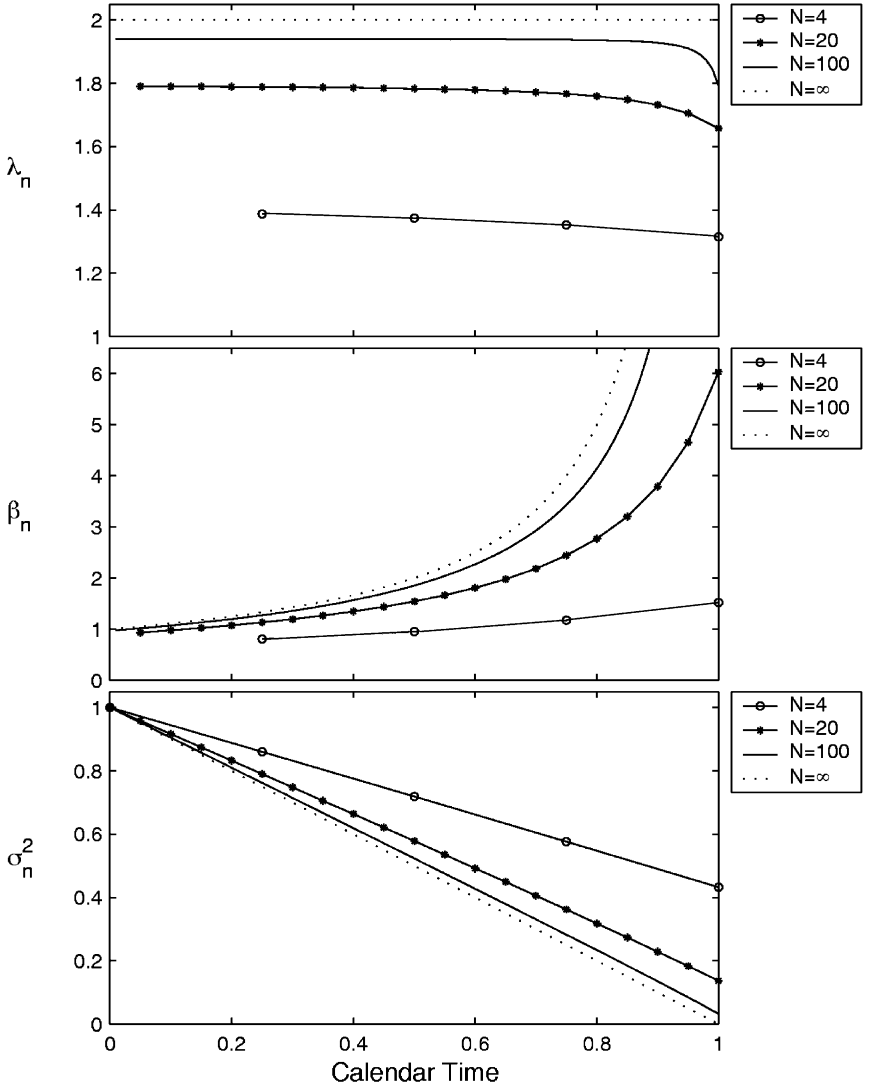

Fig. 1. Sensitivity of pricing rule λ_n, sensitivity of insider's strategy β_n, and conditional variance of the liquidation value σ_n^2 over time for different values of N, the number of auctions. Auctions take place at equally spaced intervals in [0, 1]. The number of market makers is fixed at $M = 3$. The variance of liquidity trading per unit time $\sigma_z^2 = 1$, the ex ante variance of the liquidation value $\sigma_v^2 = 1$.

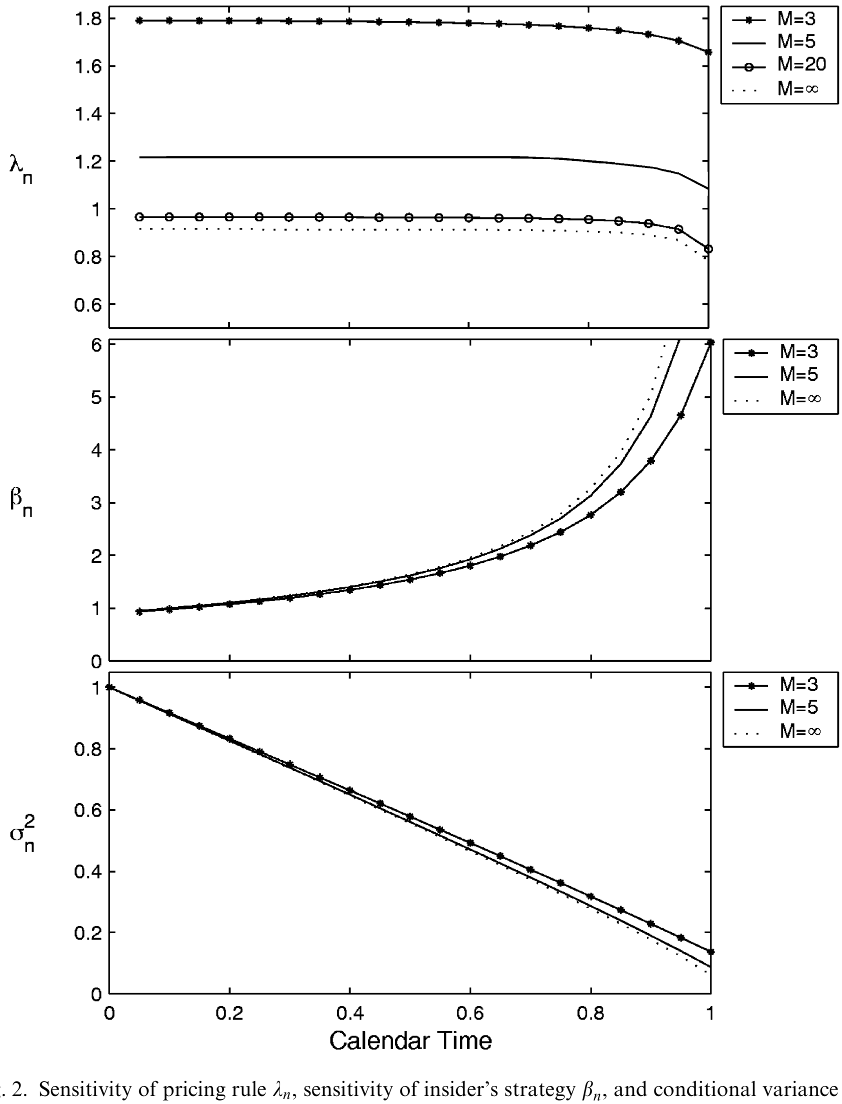

Fig. 2. Sensitivity of pricing rule λ_n, sensitivity of insider's strategy β_n, and conditional variance of the liquidation value σ_n^2 over time for different values of M, the number of market makers. The number of actions is fixed at $N = 20$. Auctions take place at equally spaced intervals in [0, 1]. The variance of liquidity trading per unit time $\sigma_z^2 = 1$, the ex ante variance of the liquidation value $\sigma_v^2 = 1$.

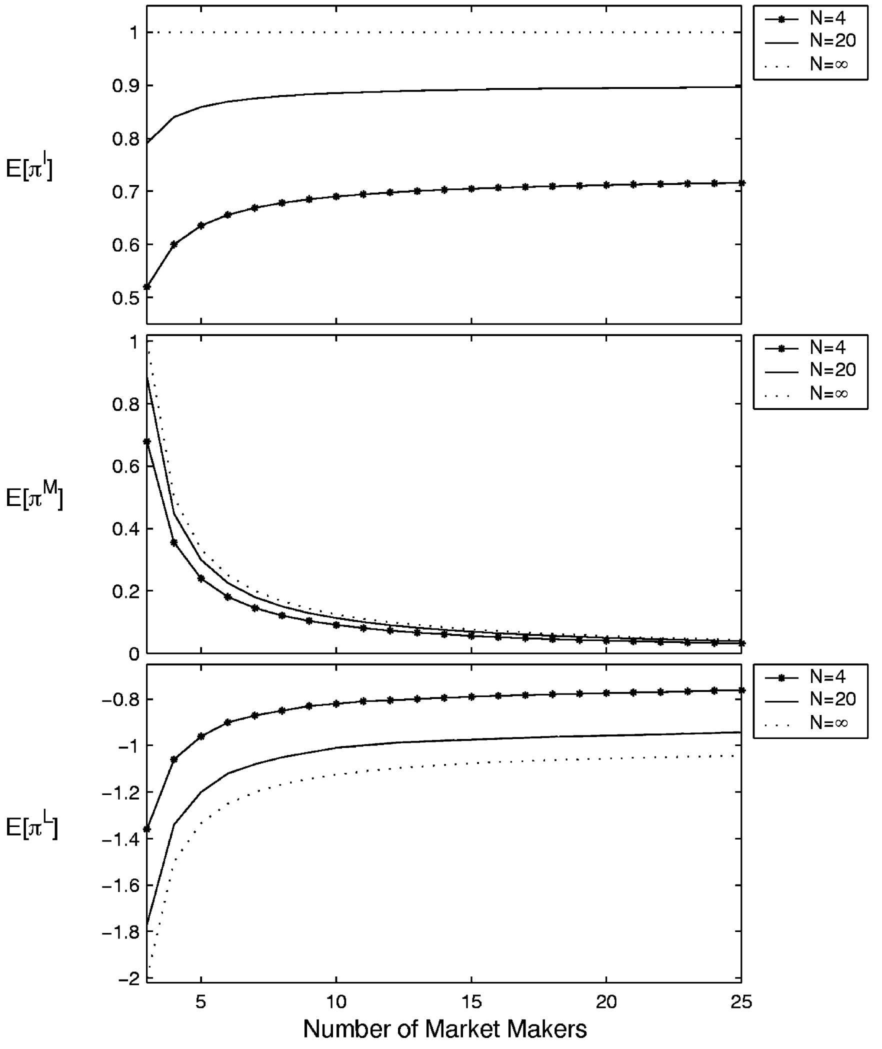

Fig. 3. Expected profits of the informed trader $E[\pi^I]$, market makers $E[\pi^M]$, and liquidity traders $E[\pi^L]$ over different values of M, the number of market makers, and for different values of N, the number of auctions. Auctions take place at equally spaced intervals in [0,1]. The variance of liquidity trading per unit time $\sigma_z^2 = 1$, the ex ante variance of the liquidation value $\sigma_v^2 = 1$.

the particular cases of $N = 4,\ 20$ and ∞. Conforming to intuition, a market with more market makers is more efficient (as measured by how much private information is incorporated in prices) and is deeper (as measured by the inverse of λ_n).[9]

3.2. Comparative statics

We now study properties of the equilibrium in Proposition 1 with respect to exogenous parameters. We are interested in how the sensitivity of the effective price schedule λ_n, the sensitivity of the informed trader's strategy β_n, conditional variance of the asset's liquidation value σ_n^2, price variance $\delta_n^2 := \mathrm{var}\{p_n \mid \mathscr{F}_{n-1}\}$, and ex ante expected profits of the informed trader $E[\pi^I]$ and of market makers $E[\pi^M]$ depend on the asset's initial uncertainty σ_v, level of liquidity trading σ_z, and auction number n.

Proposition 3 (Comparative statics). *In the linear equilibrium characterized by Proposition* 1,

1. *When σ_v increases, then $\lambda_n,\ \sigma_n,\ \delta_n,\ E[\pi^I]$, and $E[\pi^M]$ increase proportionally, and β_n decreases proportionally. When σ_z increases, then β_n, $E[\pi^I]$, and $E[\pi^M]$ increase proportionally, λ_n decreases proportionally, and σ_n and δ_n do not change.*
2. *As trading progresses (n increases), σ_n^2 decreases. Moreover, assuming that auctions take place at equally spaced intervals* (i.e., $\Delta t_n = 1/N$), *β_n and δ_n^2 increase, λ_n decreases, and σ_n^2 decreases at increasing rate with n.*

The intertemporal results in Proposition 3 provide some new insights for the classic Kyle (1985) model, which is a special case of ours. Specifically, two important findings that are often cited regarding the Kyle's sequential equilibrium are that the sensitivity of the informed trader's strategy increases and the slope of the price schedule declines over time. To our knowledge, however, these relationships have never been proved *analytically*. Instead, they are demonstrated in Kyle's original paper via numerical simulations. In those simulations, trading takes place at equally spaced auctions. Proposition 3 *proves* these claims as well as the fact that price variance δ_n^2 increases over time

[9] Note, however, that even though the *overall* market quality improves (the aggregate sensitivity λ_n decreases), the quality of *individual* markets deteriorate since an individual market maker's sensitivity $\lambda_{mn} = M\lambda_n$ increases without bound with M. This means that, with too many market makers, traders who do not split their orders across markets will incur higher costs. Moreover, since there are fixed costs C to making a market, the number of market makers must satisfy the participation constraint that the expected profit of an individual market maker is no less than C.

for the more general model. Moreover, the proposition underscores that these monotonicity results may not obtain if auctions take place at irregular intervals.

The latter is important if one believes that the level of liquidity trading varies over time in a systematic way (for example, liquidity trading may be higher after the market opens and before it closes). Indeed, recall that the amount of liquidity trading at the nth auction is determined by the time between consecutive auctions Δt_n since Δz_n is normally distributed with variance $\sigma_z^2 \Delta t_n$. Therefore, one can model time-varying liquidity trading by following one of two approaches: one can assume either that σ_z^2 is function of n and auctions are equally spaced or that σ_z^2 is constant but the intervals between auctions Δt_n are proportional to the observed level of liquidity trading.

The two approaches are equivalent, but the second one has the advantage that no changes in Proposition 1 are needed. In this case, however, t_n should be interpreted not as *physical* time but rather *transaction* time. Under such an interpretation, Proposition 3 states that λ_n, β_n, and δ_n^2 change monotonically as the announcement date approaches if the level of liquidity shocks is constant over time. However, these statistics may change nonmonotonically if the level of liquidity shocks is time-varying. This suggests that temporal patterns of liquidity trading may induce systematic trading patterns. For example, using Proposition 1 one can easily construct examples, when liquidity trading substantially decreases during certain trading periods and this causes decrease in *both* λ_n *and* β_n. (Intuitively, this happens because when the level of liquidity trading is very low, it becomes difficult for the informed trader to hide his trades and, as a result, he trades less aggressively.) This means that, contrary to common belief, λ_n and β_n may *not* always be inversely related over time.[10]

3.3. Bid–ask spreads and price dynamics

We now investigate the effect of imperfect competition of market makers on price dynamics and the bid–ask spreads. In our linear model, there are no explicit spreads. Still, we can measure the effect of the size of order flow on the asset's price by the *effective* bid–ask spread implicit in the price-quotation schedule. Given price schedule $P(\cdot)$, the effective bid–ask spread for order quantity $q > 0$ is defined as $2S(q) := P(q) - P(-q)$, the difference between the

[10] An interesting observation that follows from the proof of Proposition 3 is that the informed trading intensity becomes more monotonic as the market becomes more competitive (M increases). In fact, in the limiting case of the perfect competition, β_n is always monotonic, even in the presence of irregular liquidity trading.

price at which q shares may be bought and the price at which q shares may be sold.

In the linear equilibrium, the effective spread at the nth auction is proportional to order size q, that is $S_n(q) = \lambda_n q$. Let $S_n := S_n(1)$ denote the effective bid–ask spread for one round lot. From Proposition 1, S_n can be decomposed into two parts, component S_n^{a} due to adverse selection and component S_n^{c} due to (imperfect) competition of market makers:

$$S_n := S_n^{\mathrm{a}} + S_n^{\mathrm{c}} = \frac{\beta_n}{\rho_n^2} + \frac{1}{M-2}\frac{\beta_n}{\rho_n^2}.$$

The adverse selection component S_n^{a} is a measure of the informational content of an order flow, familiar in models of asymmetric information. It reflects how the public updates its expectation about the asset's liquidation value v^* after the arrival of order $(\Delta x_n + \Delta z_n)$. (Recall that the new expectation is revised to $v_n = v_{n-1} + \beta_n/\rho_n^2(\Delta x_n + \Delta z_n)$.) It is because of this updating that transaction prices p_n eventually converge to full information value v^*.

On the other hand, the imperfect competition component of the bid–ask spread $S_n^{\mathrm{c}} = \lambda_n - \beta_n/\rho_n^2$ reflects "frictions" in the market due to the fact that market makers post price schedules which are steeper than efficient ones. This component of the bid–ask spread represents the source of profit of market makers. The presence of component S_n^{c} induces negative serial correlation in transaction prices p_n.

The implications for the dynamics of transaction prices are formally stated in the following proposition. Denote as $\varepsilon_n := v^* - p_n$ deviation of p_n from the asset's liquidation value v^* and let $\Delta p_n = p_n - p_{n-1}$.

Proposition 4 (Price dynamics). *In the linear equilibrium*,

1. *Transaction prices do* not *follow a martingale*,

$$E[p_{n+1} \mid \mathscr{F}_n] = E[v^* \mid \mathscr{F}_n] \neq p_n.$$

2. *Price deviations ε_n are positively serially correlated for all $k \geqslant 1$*

$$E[\varepsilon_n \varepsilon_{n-k}] = (1 - \lambda_n \beta_n \Delta t_n)\sigma_{n-1}^2 > 0.$$

3. *Consecutive transaction prices are negatively correlated*

$$E[\Delta p_{n+1} \Delta p_n] = -\frac{1}{M-2}\lambda_n \beta_n \Delta t_n \sigma_{n-1}^2 < 0.$$

For all $k > 1$, $E[\Delta p_{n+k} \Delta p_n] = 0.$

In contrast to the traditional models of competitive dealer markets of Kyle (1985), Glosten and Milgrom (1985), and Easley and O'Hara (1987), in our model the martingale property of prices does not hold.[11]

Positive autocorrelation of price deviation ε_n implies that information is incorporated in prices gradually and that initial mispricing is partly carried forward. Another way to express this result is as follows. Let $e_n := v^* - v_n$ denote the market predictive error. Then one can show that the predictive errors are positively serially correlated. In particular, $E[e_n \mid e_{n-1}] = \eta_n e_{n-1}$, where $\eta_n := \sigma_n^2/\sigma_{n-1}^2$.[12] Here quantity η_n may be interpreted as a measures of the (strong form) informational efficiency of the market. Intuitively, η_n is equal to the rate of convergence of market expectation v_n to the full information value v^*. The smaller this quantity η_n, the faster the convergence. It follows from Proposition 3 that $0<\eta_n<1$, implying that the market forecast of the asset liquidation value becomes more accurate as trading progresses and as more private information is revealed by the informed trader.

Results of Proposition 4 are illustrated in Fig. 4, which plots simulated histories of market prices p_n and market expectation of the asset's liquidation value $v_n = E[v^* | \mathscr{F}_n]$ for $M = 3$ and 5. In these simulations, the number of auctions is $N = 20$, the asset initial value $v_0 = 10$, the asset's realized liquidation value is $v^* = 11$. Realization of liquidity trading is the same for both values of M. The cumulative liquidity trading $Z_n := \sum_{i=1}^n \Delta z_i$ and corresponding cumulative informed trading $X_n := \sum_{i=1}^n \Delta x_i$ are shown in the top panel of Fig. 4. From inspection of this figure it is clear that prices p_n eventually converge to v^*. Deviations of p_n from v_n are equal to the imperfect competition component of the bid–ask spread. This component causes prices p_n to "bounce" around v_n. As expected, the deviations are larger for the less competitive market with $M = 3$ than they are for the more competitive market with $M = 5$.

The next proposition summarizes cross-sectional and time-series properties of the bid–ask spreads.

Proposition 5 (Bid–ask spreads). *In the linear equilibrium,*

1. *At any time, adverse selection component S_n^{a} and imperfect competition component S_n^{c} represent constant fractions of the bid–ask spread S_n, and $S_n^{\mathrm{a}}/S_n^{\mathrm{c}} = 1/(M-2)$.*
2. *The bid–ask spread S_n is proportional to σ_v/σ_z.*

[11] Note that this does not mean that the market is not semistrong form efficient. It is easy to see that the asset expected value v_n is a martingale and that no abnormal returns can be made based on the public information.

[12] From the proof of Proposition 1, $v_n = v_{n-1} + (\beta_n/\rho_n^2)(\beta_n \Delta t_n (v^* - v_{n-1}) + \Delta z_n)$. Therefore, $E[e_n \mid e_{n-1}] = (\sigma_n^2/\sigma_{n-1}^2)\, e_{n-1}$.

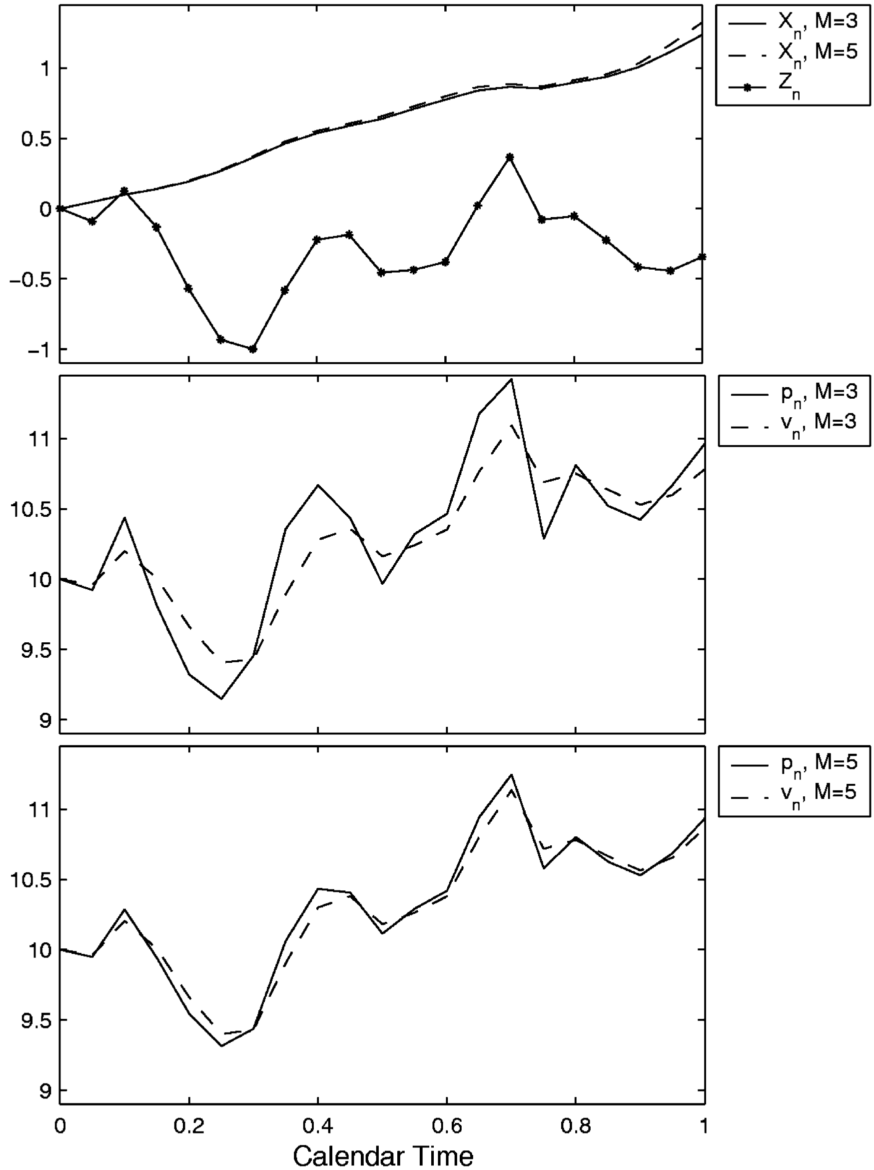

Fig. 4. Simulated price dynamics. The first panel plots cumulative informed trading X_n and liquidity trading Z_n over time. The second and third panels plot market price p_n and expectation of the liquidation value $v_n = E[v^* \mid \mathscr{F}_n]$ for the number of market makers $M = 3$ and 5. The number of actions $N = 20$. Auctions take place at equally spaced intervals in [0, 1]. The variance of liquidity trading per unit time $\sigma_z^2 = 1$, the ex ante mean and variance of the liquidation value are $v_0 = 10$ and $\sigma_v^2 = 1$. The asset's liquidation value $v^* = 11$.

3. *Assuming that auctions take place at equally spaced intervals* ($\Delta t_n = 1/N$), *then* S_n *monotonically decreases with n.*

3.4. Empirical issues

Propositions 4 and 5 have important implications for empirical work on the determinants of the bid–ask spreads.

First, a number of recent empirical studies estimate the effective bid–ask spread and determine components of the spread due to adverse selection, inventory-holding costs, and order processing costs. See, for example, Glosten and Harris (1988), Stoll (1989), George et al. (1991), and, more recently, Huang and Stoll (1997), and Madhavan et al. (1997). These studies analyze different datasets of transaction data for different exchanges and rely upon different statistical models. They all conclude that the permanent adverse selection component is indeed present in the data but provide varying estimates for the bid–ask spread components.

Proposition 5 suggests that an empiricist analyzing the bid–ask spreads needs also account for a fourth component, which is due to imperfect competition. Usually this component will vary across different markets, and, therefore, it may at least partially reconcile the differences in the estimates of the above studies. Moreover, from Proposition 5 the imperfect competition component S_n^c is proportional to the adverse selection component S_n^a implying that the number of market makers M is an important cross-sectional determinant of the bid–ask spread.[13] Empirical evidence on how competition among dealers affects the bid–ask spread has recently been reported in Huang and Masulis (1999) who study the spot foreign exchange market. They find that the number of dealers active in the market is an important determinant of the bid–ask spreads and estimate that addition of one more competing dealer lowers the average quoted spread by 1.7%.[14]

Second, in view of Proposition 5 the size of the bid–ask spreads is proportional to σ_v/σ_z. This means that spreads are wider when the adverse selection problem is more severe (i.e., when asset value uncertainty is large, and

[13] Recall that M may also be interpreted as the number of exchanges where a security is traded.

[14] Observe that

$$\frac{\Delta S_n}{S_n \Delta M} \approx -\frac{1}{(M-1)(M-2)}.$$

By taking the RHS of the above equation equal to −1.7%, we obtain a back-of-the-envelope estimate of the average number of market makers in the foreign exchange market, $M \approx 9$. This estimate is in line with the actual figures reported in Huang and Masulis: the average number of dealers in the market is 16, and the average number of large dealers (whose competition is found to have the strongest impact on the bid–ask spreads) is 4.

liquidity trading is small). These relationships can be tested for a cross-section of stocks, and they are in agreement with empirical evidence. For example, the recent study by Chung and Charoenwong (1998) finds that spreads are positively associated with asset risk (which is positively related to σ_v), negatively associated with trading volume, firm size, and share price (they are all negatively correlated with σ_z) and the number of exchange listings (a proxy for M).

Third, for a time-series analysis, our model suggests that the bid–ask spreads as well as degree of adverse selection decline as the announcement date approaches.[15] Empirical study by Madhavan et al. (1997) finds that information asymmetry indeed decreases steadily throughout the day. McInish and Wood (1992) analyze intraday the bid–ask spreads for the NYSE and find that they have a reverse J-shape. That is, spreads first rapidly decline after market opening, after which they remain stable and slightly increase before the market close.[16]

Another somewhat unexpected prediction of Proposition 3 is that price variance δ_n^2 increases over time. This implies that, in a time-series analysis, the size of the bid–ask spread and price volatility are *negatively* related. This result is in contrast to the conclusion of the static models (see, for example, Madhavan (1992)) and the rationale for it is as follows. As trading progresses, information asymmetries in the market decrease. This has the opposite effects on price variance: (1) the sensitivity of the price schedule λ_n (and, consequently, the bid–ask spread) declines and (2) the informed trader trades more aggressively (the sensitivity of his strategy β_n increases). It can be shown that the second effect dominates and price variance increases over time.[17]

Finally, there is an important policy implication that follows from Proposition 4. Intuitively, this proposition identifies two distinct effects in the dynamics of transaction prices. First, due to the permanent adverse selection component, there is a ''persistency'' effect, in the sense that price changes are *not* independent ex post. Because prices eventually converge to the liquidation value, price deviations ε_n are positively autocorrelated. Second, due

[15] Easley and O'Hara (1992) and Madhavan (1992) derive similar relationships in different theoretical models.

[16] Larger spreads near to the market close may be due to overnight inventory costs of market makers.

[17] In the limiting case of continuous trading which is discussed in the next subsection, the sensitivity of the price schedule λ_t is constant, and the sensitivity of the informed trader's strategy β_t increases as $1/(1-t)$ over time.

It is worth mentioning, however, that numerical simulations suggest that negative time-series association between the size of the bid–ask spread and price variance may be very weak. From the proof of Proposition 3, the fact that price variance increases over time is equivalent to the fact that σ_n^2 is concave. Visual inspection of Figs. 1 and 2 indicates, however, that the plot of σ_n^2 is virtually a straight line.

to the imperfect competition component of the bid–ask spread, there is also a "bounce" effect which causes consecutive price changes to be negatively correlated. Furthermore, a simple relationship exists between the two effects that can be empirically tested on transaction data. Observe that price change Δp_n may be written as

$$p_n - p_{n-1} = \frac{M-1}{M-2}\frac{\beta_n}{\rho_n^2}(\Delta x_n + \Delta z_n) - \frac{1}{M-2}\frac{\beta_{n-1}}{\rho_{n-1}^2}(\Delta x_{n-1} + \Delta z_{n-1}). \quad (14)$$

From the last equation, Δp_n is positively related to the *current* order $(\Delta x_n + \Delta z_n)$ (due to adverse selection) and is negatively related to the *previous* order $(\Delta x_{n-1} + \Delta z_{n-1})$ (due to imperfect competition). Suppose now that an econometrist is able to observe transactions prices p_t and trading volumes $(\Delta x_t + \Delta z_t)$ where t indexes trades within the trading day. The econometrist then can estimate the following statistical model

$$p_t^i - p_{t-1}^i = \alpha_t q_t^i - \gamma_t q_{t-1}^i + \xi_t^i, \quad (15)$$

where p_{t-1}^i and p_t^i are consecutive transaction prices, q_{t-1}^i and q_t^i are corresponding order sizes, superscript i indexes trading days, and ξ_t^i is a random variable with zero mean which captures the effect of stochastic rounding errors associated with price discreteness and other market imperfections. Here coefficients α_t and γ_t are constants estimated separately for, say, each of 15-min intervals of a day. For short intervals, $\beta_n/\rho_n^2 \approx \beta_{n-1}/\rho_{n-1}^2$ and thus from Eq. (14) we obtain the testable restriction that, for all intervals t,

$$\frac{\alpha_t}{\gamma_t} = M - 1.^{18} \quad (16)$$

Let us define $\phi_t := \alpha_t/\gamma_t + 1$ as an empirical measure of market competitiveness. Intuitively, one may interpret quantity ϕ_t as the *effective* number of liquidity suppliers in the market consistent with observed transaction prices. In particular, ϕ_t must be large for highly competitive markets. Characterization of different markets based on this measure is an interesting direction for future empirical research.

Statistical model (15) and (16) has an important implication for policy makers and regulators. Consider as example the recent controversy over the spreads in the Nasdaq market. In October 1994 the Justice Department and the Security and Exchange Commission started an investigation regarding possible collusion among dealers to maintain spreads at artificially high levels. The investigation stemmed directly from studies by Christie and Schultz (1994) and Christie et al. (1994). By analyzing pricing patterns, they conclude that the

[18] In the continuous equilibrium discussed in the next section, an analog of restriction in (16) holds precisely.

Nasdaq dealers tacitly cooperated to increase the bid–ask spreads by avoiding odd-eights quotes. The two studies, however, do not demonstrate that the spreads were, in some sense, too wide and that the dealers' profits were, in some sense, too high. In the context of our model, we can address these issues by checking whether the ratio of the imperfect competition component S^c to the adverse selection component S^a was too large in the NASDAQ market. Specifically, if one estimates statistical model (15) and (16) for a market with M market makers and finds that $\phi_t \ll M$, this could indicate that the actual profits of market makers are significantly higher than the "normal" profits predicted by the model under the assumption that market makers act *noncooperatively*. Such a conclusion, therefore, would be consistent with a possible collusion of market makers.[19]

3.5. Continuous auction, $N = \infty$

This section investigates the properties of the equilibrium when intervals between auctions become very small. Let $\tau := \{t_1, \ldots, t_{N-1}\}$ denote the partition of the interval [0, 1] which corresponds to the sequence of auction dates. The maximum interval between auctions is denoted $\Delta\tau$:

$$\Delta\tau := \max_n \Delta t_n.$$

For any partition τ, let $\beta^\tau(t)$, $\lambda^\tau(t)$, $a^\tau(t)$, $b^\tau(t)$, $c^\tau(t)$, and $\rho^\tau(t)$ be defined as continuous on [0, 1] functions which are (i) linear on each interval $[t_{n-1}, t_n]$, and (ii) for all t_n satisfy conditions $\beta^\tau(t) = \beta_n$, $\lambda^\tau(t_n) = \lambda_n$, etc., where constants λ_n, β_n, a_n, b_n, c_n, and ρ_n are given in Proposition 1. The next proposition states that, as intervals between auctions approach zero ($\Delta\tau \to 0$), functions $\beta^\tau(t)$, $\lambda^\tau(t)$, $a^\tau(t)$, $b^\tau(t)$, $c^\tau(t)$, and $\rho^\tau(t)$ converge to continuously differentiable functions on [0, 1).

Proposition 6 (Continuous market). *Holding ρ_0 constant, consider a sequence of partitions $\{\tau_k\}_{k=1}^{\infty}$ such that $\Delta\tau_k \to 0$. Then sequences of functions $\beta^{\tau_k} \times (t)$, $\lambda^{\tau_k}(t)$, $a^{\tau_k}(t)$, $b^{\tau_k}(t)$, $c^{\tau_k}(t)$, and $\rho^{\tau_k}(t)$, $k = 1, 2, \ldots$, converge on* [0, 1) *to functions $\beta(t)$, $\lambda(t)$, $a(t)$, $b(t)$, $c(t)$, and $\rho(t)$, where*

$$\lambda(t) = \frac{M-1}{M-2}\frac{1}{\rho_0}, \tag{17}$$

$$\beta(t) = \frac{\rho_0}{1-t}, \tag{18}$$

[19] See also Dutta and Madhavan (1997) who develop a theoretical model that distinguishes between noncooperative competition and explicit collusion in dynamic dealer markets without asymmetric information.

$$a(t) = \tfrac{1}{2}\rho_0, \tag{19}$$

$$b(t) = \frac{\sigma_z^2}{2\rho_0}(1-t), \tag{20}$$

$$c(t) = \frac{1}{M(M-2)}\frac{\sigma_z^2}{2\rho_0}(1-t), \tag{21}$$

$$\rho^2(t) = \frac{\rho_0^2}{1-t}. \tag{22}$$

The convergence is uniform on $[0, t']$ *for all* $t' < 1$.

Proposition 6 can be interpreted as follows. When the interval between auctions goes to zero, the unique linear equilibrium of the sequential auction model given in Proposition 1 approaches a linear equilibrium of a continuous market. In this continuous equilibrium, for $m = 1, \ldots, M$ and $t \in [0, 1)$

$$\mathrm{d}z_m(t) = \frac{1}{M}\,\mathrm{d}z(t),$$

$$\mathrm{d}x_m(t) = \frac{\beta(t)}{M}(v^* - v(t))\,\mathrm{d}t,$$

$$p_m(t) = v(t) + \lambda(t)M(\mathrm{d}x_m(t) + \mathrm{d}z_m(t)),$$

$$E[\pi^I(t)|\mathscr{F}(t), v^*] = a(t)(v^* - v(t))^2 + b(t),$$

$$E[\pi_m^M(t)|\mathscr{F}(t)] = c(t),$$

where at time t, $\mathrm{d}x_m(t)$ and $\mathrm{d}z_m(t)$ are orders of the informed and liquidity traders to market maker-m; $p_m(t)$ is the price at which these orders are executed, $\pi^I(t)$ and $\pi_m^M(t)$ are profits of the informed trader and market maker-m on positions acquired on $[t, 1]$; $\mathscr{F}(t) = \{\{p_m(h)\}_{m=1}^M \mid 0 \leqslant h < t\}$ is public information at time t; $v(t) := E[v^* | \mathscr{F}(t)]$, and $\sigma^2(t) := \mathrm{var}\{v^* \mid \mathscr{F}(t)\} = \sigma_z^2/\rho^2(t)$.

Proposition 6 states that, when the time between auctions goes to zero, equilibrium outcomes with a finite number of market makers become very similar to those reported in Kyle (1985) for the perfect competition case ($M = \infty$). In both cases, the informed trader follows the same strategy and earns the same expected (ex ante) profit of $\sigma_z\sigma_v$. The asset's conditional variance linearly decreases from σ_v^2 to zero (i.e., private information is incorporated into prices at a constant rate)

$$\sigma^2(t) = \sigma_v^2(1-t).$$

By the end of trading all private information is incorporated into prices. Also, in both cases, market depth is constant over time, however, depth is lower for the imperfect competition case reflecting the fact that market makers post steeper than efficient price schedules. Expected profit of all market makers is positive, and this results in larger expected loss for liquidity traders:

$$E[\pi^M] = \frac{1}{2(M-2)}\sigma_z\sigma_v, \quad E[\pi^L] = -\left(1 + \frac{1}{2(M-2)}\right)\sigma_z\sigma_v.$$

Kyle (1985) uses three measures to characterize different aspects of market liquidity: (i) tightness, defined as the cost of turning around a position over a short period of time, (ii) depth, defined as the size of an order flow necessary to move prices a unit amount and (iii) resiliency, defined as the speed with which prices recover from an uninformative shock.

In the case of perfect competition of Kyle (1985), the market is infinitely tight because it is costless to turn over a position very quickly. It has a constant finite depth (measured by $1/\lambda(t)$). Market resiliency, which is determined by the trading of the informed trader, increases over time with the market becoming infinitely resilient near the end of the trading.

It is clear from Proposition 6 that imperfect competition changes properties of continuous markets very little. The imperfect competition market is characterized by the same resiliency. It has a constant but lower depth. The market is also infinitely tight. As with perfect competition, the reason for this is that given infinitely many trading opportunities, the informed trader trades on his residual private information at the same rate at each moment. The informed trader, effectively, acts as a perfectly discriminating monopsonist who moves along the residual supply curve. Note, however, that in a sequential equilibrium when trading takes place at discrete auctions, for both the perfect and imperfect competition cases, the markets are *not* infinitely tight because the informed trader faces an upward sloping supply curve and has only a finite number of trading opportunities. Moreover, the cost of turning over a position is higher under imperfect competition.

A continuous time version of Kyle (1985) is considered in Back (1992). Back allows for a more general class of distributions for asset value and participants' strategies and provides a more formal analysis of trading in a continuous market. It is interesting to see whether Back's analysis can be extended to the case of imperfect competition of market makers. One observation suggests that such an extension may not be straightforward: in the continuous equilibrium (A.1)-(A.1), price $p(t) = p_m(t)$ is *not* a Markov process. Because of implicit bid–ask bounce, price $p(t + \mathrm{d}t)$ at time $t + \mathrm{d}t$ depends not only on the current price $p(t)$ and order flow $\mathrm{d}x(t) + \mathrm{d}z(t)$ but also on the price one instant before $p(t - \mathrm{d}t)$.

4. Discussion

Up to this point the number of market maker has been assumed fixed and exogenously given. One can also use the model to determine the equilibrium number of market makers M^* when there are fixed costs $C>0$ to making a market. With no barriers to entry, new market makers will enter the market as long as the expected profit of an individual market maker given in Proposition 1, $c_0(M^*)$, is greater than C. From comparative statics in Proposition 3, the expected profit on market makers is proportional to both σ_v and σ_z. Therefore, for given fixed costs C, the equilibrium number of market makers M^* is positively related to both the asset uncertainty and the level of liquidity trading. This can be verified empirically in a cross-sectional analysis of different stocks.[20]

We also wish to discuss two important aspects of the informed trading in the market. The first aspect is well understood in the microstructure literature. It concerns a positive contribution of the informed trader to price discovery. The informed trader, in effect, creates a public good by making a market more informationally efficient. He determines the resiliency of the market and causes transaction prices to eventually converge to the asset's fundamental value despite stochastic demands of liquidity traders.

The second aspect is less-recognized but interesting nevertheless. To illustrate it, consider again the market without the informed trader discussed in Section 2.2. In this market, when $M \geqslant 3$ there is a unique linear equilibrium with perfectly competitive outcome $\lambda_n = 0$ for all n. With no asymmetric information, the market depth is infinite; the bid–ask spread is zero as are market makers' profits. The asset's price remains v_0 until the public announcement, at which point the price jumps to v^*, meaning that the market is trivially strong form efficient.

Suppose now that there are fixed costs $C>0$ associated with making a market. Then, no matter how small these costs, no market maker will be willing to stand by to provide liquidity, and the market will cease to function. Without the informed trader it is impossible for market makers to sustain expected profits above C. In other words, it is the presence of the informed trader that makes the market viable. Since the source of market makers' profit is the imperfect competition component S_n^{c} of the spread, which is proportional to the adverse selection component S_n^{a}, market makers actually *prefer* more information asymmetries to less. Moreover, some minimal amount of the informed trading is absolutely *necessary* for market makers to be able to break even. These somewhat surprising conclusions are in sharp contrast to Glosten and Milgrom (1985), Madhavan (1992), and related

[20] It is plausible, however, that market makers' cost C may also depend on σ_v and σ_z.

models, in which markets fail when asymmetric information problems become too severe.[21]

5. Conclusion

This paper develops a dynamic microstructure model of liquidity provision in which M strategic market makers compete in price schedules to supply liquidity to informed and uninformed traders. When $M > 2$, there is a unique linear equilibrium which is symmetric and characterized by simple recursive strategies of the market participants. In the equilibrium, market makers post price schedules which are steeper than efficient ones. In the limiting case when M approaches infinity, the equilibrium converges to the zero-profit sequential equilibrium in Kyle (1985).

When the number of market makers is finite, the bid–ask spreads can be decomposed into two components, one due to adverse selection and the other due to imperfect competition. At any time, the two components are proportional to each other with a coefficient of proportionality depending on the number of market makers M. Because of the adverse selection, deviations of prices from the ex post liquidation value are positively correlated so that prices eventually incorporate all the informed trader's private information. The presence of the imperfect competition component induces negative serial correlation in transaction prices. Based on theoretical properties of the bid–ask spreads, several testable empirical hypothesis are derived. In particular, our model suggest a new empirical measure of market competitiveness which can be estimated from the history of transaction prices and trading volumes. The measure may be interpreted as the effective number of competing liquidity providers consistent with the adverse selection and imperfect competition components of the bid–ask spread observed in the market. Comparison of different markets based on this measure can be an interesting direction for future empirical research.

When the intervals between auctions go to zero, the sequential auction approaches continuous trading. Compared with the continuous market in Kyle (1985), the paper finds that the continuous market with a finite number of market makers is also infinitely tight; it has the same resiliency but lower market depth.

[21] It is important to emphasize that the analysis in this paper has been restricted to *linear* equilibria only and that some of our conclusions may not hold when more general strategies are allowed. In particular, the conclusion that in many microstructure models of asymmetric information the market fails when the asymmetry is too severe may not necessarily generalize if market makers can post nonlinear price schedules. Bhattacharya and Spiegel (1991) is the first paper to consider nonlinear price schedules which are twice continuously differentiable, while Bhattacharya et al. (1995) allow for completely arbitrary price schedules. Nonlinear equilibria are also investigated in Glosten (1994) and Biais et al. (2000).

In this paper, we study strategic aspects of market makers' trading behavior by focusing on the classic Kyle's environment. However, the basic model can easily be extended to accommodate richer market structures. Such extensions may include (1) multiple informed traders (as in Foster and Viswanathan (1993) and Holden and Subrahmanyam (1992)), (2) price-sensitive demands of liquidity traders (as in Bernhardt and Hughson (1997)) and (3) time-varying variance of liquidity demands.

Appendix

Proof of Proposition 1. The proof is divided into two steps. In the first step, a backward induction is used to simultaneously prove that, in each auction, (i) liquidity traders, the informed trader and market makers use symmetric, recursively defined strategies given in (2)–(4); (ii) expected profit functions of the informed trader and market makers are as given in (5) and (6) and (iii) constants β_n, λ_n, ρ_n, a_n, b_n, and c_n satisfy Eqs. (7)–(12). The second step proves that the difference equation system derived in the first step characterizes the unique linear equilibrium.

Step 1: We proceed by a backward induction. Suppose that there are constants a_n, b_n, and c_n such that

$$E[\pi^I_{n+1} \mid \mathscr{F}_n, v^*] = a_n(v^* - v_n)^2 + b_n,$$

$$E[\pi^M_{m(n+1)} \mid \mathscr{F}_n] = c_n.$$

The inductive hypothesis holds for $n = N$, with $a_N = b_N = c_N = 0$ (since no profits can be made after the last auction).

Without loss of generality, participants' strategies at the nth auction can be represented as

$$\Delta z_{mn} = \gamma_{mn}\Delta z_n + u_m \quad \text{such that} \quad \sum_{m=1}^{M} \Delta z_{mn} = \Delta z_n,$$

$$\Delta x_{mn} = \beta_{mn}(v^* - v_{n-1})\Delta t_n + v_m,$$

$$p_{mn} = v_{n-1} + \lambda_{mn}(\Delta x_{mn} + \Delta z_{mn} + w_m),$$

where γ_{mn}, β_{mn}, and λ_{mn} are constants, and u_m, v_m, and w_m are some functions of $\mathscr{F}_{n-1}$, the public information at time $t-1$. The following lemma (which is proved later) states that, in a linear equilibrium, u_m, v_m and w_m must be zero.

Lemma 1. Under the inductive hypothesis,

$$\Delta z_{mn} = \gamma_{mn}\Delta z_n \quad \textit{such that} \quad \sum_{m=1}^{M} \Delta z_{mn} = \Delta z_n,$$

$$\Delta x_{mn} = \beta_{mn}(v^* - v_{n-1})\Delta t_n,$$

$$p_{mn} = v_{n-1} + \lambda_{mn}(\Delta x_{mn} + \Delta z_{mn}).$$

Consider now optimal responses of the market participants who face strategies of other agents as given in Lemma 1. We first show that given slopes of price schedules $\lambda_{1n}, \ldots, \lambda_{Mn}$, the liquidity and informed traders choose their sensitivities γ_{mn} and β_{mn} proportional to $1/\lambda_{mn}$. (Recall from 1 that β_n denotes the *aggregate* sensitivity of the informed trader's strategy and λ_n denotes the slope of the *effective* price schedule provided by M market makers.)

A. *Liquidity traders' optimization problem*:

$$\max_{\{\Delta z_{mn}\}_{m=1}^{M}} E\left[\sum_{m=1}^{M}(v^* - p_{mn})\Delta z_{mn} \mid \mathscr{F}_{n-1}\right] \quad \text{such that} \quad \sum_{m=1}^{M}\Delta z_{mn} = \Delta z_n. \tag{A.1}$$

From Lemma 1 the following three conditions hold

$$(v^* - p_{mn})\Delta z_{mn} = (v^* - v_{n-1})(1 - \lambda_{mn}\beta_{mn}\Delta t_n)\Delta z_{mn} - \lambda_{mn}\Delta z_{mn}^2,$$

$$\Delta z_{mn} = \gamma_{mn}\Delta z_n,$$

$$E[(v^* - v_{n-1})\Delta z_n \mid \mathscr{F}_{n-1}] = 0$$

and we obtain that solution to the constrained optimization problem is

$$\Delta z_{mn} = \frac{\lambda_n}{\lambda_{mn}}\Delta z_n.$$

The second order conditions are simply $\lambda_{mn} > 0$.

B. *The informed trader's optimization problem*:

$$\max_{\{\Delta x_{mn}\}_{m=1}^{M}} E\left[\sum_{m=1}^{M}(v^* - p_{mn})\Delta x_{mn} + a_n(v^* - v_n)^2 + b_n \mid \mathscr{F}_{n-1}, v^*\right]. \tag{A.2}$$

When choosing how much to trade with each market maker, the informed trader takes into account the effect his trades have on prices in both the current and future auctions. In particular, the informed trader needs to evaluate the expectation v_n of the liquidation value conditional on public information. After the nth auction, the public observes prices p_{mn} and order flows

$$\Delta x_{mn} + \Delta z_{mn} = \beta_{mn}(v^* - v_{n-1})\Delta t_n + \frac{\lambda_n}{\lambda_{mn}}\Delta z_n.$$

From normality of v^* and Δz_n we obtain that

$$v_n = E[v^* \mid \mathscr{F}_n] = \begin{cases} v_{n-1} + (\beta_n/\rho_n^2)(\Delta x_n + \Delta z_n) & \text{if } \beta_{mn} = \beta_n\lambda_n/\lambda_{mn} \\ v^* & \text{otherwise} \end{cases}$$

and

$$\sigma_n^2 = E[(v^* - v_n)^2 \mid \mathscr{F}_n] = \begin{cases} \sigma_z^2 \sigma_{n-1}^2 / (\sigma_z^2 + \sigma_{n-1}^2 \beta_n^2 \Delta t_n) & \text{if } \beta_{mn} = \beta_n \lambda_n / \lambda_{mn} \\ 0 & \text{otherwise.} \end{cases}$$

The last two equations are written in the form that underscores the fact that there is discontinuity in the expectation v_n corresponding to two different types of the informed trader's strategies. Specifically, the informed trader can choose to either "imitate" liquidity trading by submitting orders proportional to $1/\lambda_{mn}$ (i.e., $\Delta x_{mn} = (\lambda_n/\lambda_{mn})\Delta x_n$) or "reveal" his private information to the public otherwise. In the first case, the total informed and uninformed order to market maker-m is also proportional to $1/\lambda_{mn}$. Therefore, order flows to all market makers are perfectly correlated and all trades are executed at the same price p_n:

$$p_n := p_{1n} = \cdots = p_{Mn} = v_{n-1} + \lambda_n(\Delta x_n + \Delta z_n).$$[22]

In contrast, in the second case, not all order flows are perfectly correlated and realized prices of at least two market makers will be different. This allows the public to disentangle the informed trading from the liquidity trading and to infer the asset liquidation value. It is easy to see, however, that it is never optimal for the informed trader to "reveal". Assume to the contrary that he does "reveal" at the nth auction. This implies that $\pi_{n+1}^I = 0$ and the informed trader maximizes his profit from the nth auction

$$\max_{\{\Delta x_{mn}\}_{m=1}^M} E\left[\sum_{m=1}^M (v^* - p_{mn})\Delta x_{mn} \mid \mathscr{F}_{n-1}, v^*\right].$$

The solution to this problem is

$$\Delta x_{mn} = \frac{\lambda_n}{\lambda_{mn}} \Delta x_n. \tag{A.3}$$

In other words, even in this case, it is optimal for the informed trader to submit orders proportional to $1/\lambda_{mn}$.[23]

[22] Note that for market maker-m, observing his own order flow $(\Delta x_{mn} + \Delta z_{mn})$ is informationally equivalent to observing the aggregate order flow $(\Delta x_n + \Delta z_n)$.

[23] The conclusion that the informed trader never "reveals" can also be derived from a no-arbitrage argument: In a market equilibrium, all market makers must set the same price because otherwise a marginal order can be moved to the dealer with a lower price.

Since the informed trader submits orders as in (A.3), his optimization problem becomes:

$$\max_{\Delta x_n} E\left[\sum_{m=1}^{M}(v^* - v_{n-1} - \lambda_n(\Delta x_n + \Delta z_n))\frac{\lambda_n}{\lambda_{mn}}\Delta x_n \right.$$
$$\left. + a_n\left(v - v_{n-1} - \frac{\beta_n}{\rho_n^2}(\Delta x_n + \Delta z_n)\right)^2 + b_n \mid \mathscr{F}_{n-1}, v^*\right]$$
$$= \max_{\Delta x_n}\left\{(v^* - v_{n-1} - \lambda_n \Delta x_n)\Delta x_n + a_n\left(v^* - v_{n-1} - \frac{\beta_n}{\rho_n^2}\Delta x_n\right)^2 \right.$$
$$\left. + a_n\frac{\beta_n^2}{\rho_n^4}\sigma_z^2\Delta t_n + b_n\right\}.$$

Solving the above maximization problem yields the informed trader's optimal strategy

$$\Delta x_n = \frac{1 - 2a_n\beta_n/\rho_n^2}{2\lambda_n - 2a_n\beta_n^2/\rho_n^4}(v^* - v_{n-1}).$$

This proves (8). It can also be verified that the second order condition is given by (13) and that equations for constants a_{n-1} and b_{n-1} are given by (9) and (10).

C. *Market maker-m knows the optimal strategies of the informed and liquidity traders and he solves the following optimization problem*:

$$\max_{\lambda_{mn}} E[(p_{mn} - v^*)(\Delta x_{mn} + \Delta z_{mn}) + c_n \mid \mathscr{F}_{n-1}]$$
$$= \max_{\lambda_{mn}} E\Big[((v^* - v_{n-1})(\lambda_n\beta_n\Delta t_n - 1) + \lambda_n\Delta z_n)$$
$$(\beta_n\Delta t_n(v^* - v_{n-1}) + \Delta z_n)\frac{\lambda_n}{\lambda_{mn}} + c_n \mid \mathscr{F}_{n-1}\Big]$$
$$= \max_{\lambda_{mn}}\left\{\sigma_{n-1}^2(\lambda_n\beta_n\Delta t_n - 1)\frac{\lambda_n}{\lambda_{mn}}\beta_n\Delta t_n + \frac{\lambda_n^2}{\lambda_{mn}}\sigma_z^2\Delta t_n + c_n\right\}. \tag{A.4}$$

After simplification, the first order condition becomes

$$\rho_n^2\lambda_n(\lambda_{mn} - 2\lambda_n) - \beta_n(\lambda_{mn} - \lambda_n) = 0, \tag{A.5}$$

where we have used condition $\lambda_{mn} > 0$, relationships $\rho_n^2 = \rho_{n-1}^2 + \beta_n^2\Delta t_n$ and

$$\frac{\partial\lambda_n}{\partial\lambda_{mn}} = \frac{\lambda_n^2}{\lambda_{mn}^2}.$$

It follows from Eq. (A.5) that λ_{mn} must be the same for all market makers:

$$\lambda_{mn} = \lambda_n\frac{2\lambda_n\rho_n^2 - \beta_n}{\lambda_n\rho_n^2 - \beta_n} = M\lambda_n.$$

Therefore, for $M > 2$

$$\lambda_n = \frac{M-1}{M-2}\frac{\beta_n}{\rho_n^2}.$$

Finally, it can be verified that the second order conditions are always satisfied, and that the equation for constant c_{n-1} is as in (11).[24]

Step 2: The proof that the difference equation system in Proposition 1 is always solvable and describes the unique linear equilibrium is a straightforward modification of Kyle's (1985) proof for the case of perfect competition (see Kyle (1985, pp. 1325–1326)) and is omitted. □

Proof of Lemma 1. Assume that the participants' strategies are given as in (A.1)–(A.6). Consider first the optimization problem for liquidity traders (A.6). It is easy to show that, for $m = 1, \dots, M$, the first order conditions with respect to u_m are

$$2u_m + v_m + w_m = \frac{\lambda_n}{\lambda_{mn}} \sum_{m=1}^{M} (v_m + w_m), \tag{A.7}$$

where λ_n is defined in (1).

[24] *Remark*. In the above analysis equilibrium strategies of market makers-*m*, we essentially set to zero the *partial* derivative

$$\frac{\partial E[\pi_{mn}^M \mid \mathcal{F}_{n-1}]}{\partial \lambda_{mn}}$$

of the expected profit from the remaining auctions with respect to sensitivity λ_{mn} as opposed to the *complete* derivative

$$\frac{dE[\pi_{mn}^M \mid \mathcal{F}_{n-1}]}{d\lambda_{mn}} = \frac{\partial E[\pi_{mn}^M \mid \mathcal{F}_{n-1}]}{\partial \lambda_{mn}} + \frac{\partial E[\pi_{mn}^M \mid \mathcal{F}_{n-1}]}{\partial \beta_n} \frac{\partial \beta_n}{\partial \lambda_{mn}}.$$

In the latter case, market maker-*m*, who at each auction moves ahead of the informed trader, takes into account the dependence of the informed trader's sensitivity β_n on observed λ_{mn}. (This dependence, in particular, is apparent from Eq. (8).) For example, if market makers post very steep price schedules at the *n*th auction ($n < N$), this causes the informed trader to trade less aggressively in the current auction and thus increases asymmetric information in the future auctions. Analytical dependence of β_n on sensitivities λ_{mn} is rather complicated, but, fortunately, we do not have to worry about it because in our model

$$\frac{\partial E[\pi_{mn}^M \mid \mathcal{F}_{n-1}]}{\partial \beta_n} = \frac{\partial E[\pi_{mn}^I \mid \mathcal{F}_{n-1}]}{\partial \beta_n} = 0. \tag{A.6}$$

To see why equalities in (A.6) hold, recall that, in market-*m*, the profits of the market maker and the informed trader are always equal to the loss of liquidity traders, that is,

$$\pi_{mn}^M + \pi_{mn}^I = \frac{\lambda_n^2}{\lambda_{mn}} \Delta z_n^2.$$

In the above equation, the RHS does not depend on the informed trader's sensitivity β_n, and since the informed trader chooses β_n optimally (i.e., by maximizing his expected profit), we obtain equalities in (A.6).

In the optimization problem of the informed trader (A.2), compute the expectation of the first order conditions for v_m conditional on public information $\mathscr{F}_{n-1}$ to obtain

$$u_m + 2v_m + w_m = 0. \tag{A.8}$$

Define $W := \sum_{m=1}^{M} w_m$. From (A.7) and (A.8) we have that $\sum_{m=1}^{M} u_m = 0$ and $\sum_{m=1}^{M} v_m = -\frac{1}{2}W$. Furthermore,

$$u_m + v_m = -\frac{2}{3}w_m + \frac{1}{6}\frac{\lambda_n}{\lambda_{mn}}W. \tag{A.9}$$

Finally, in the optimization problem (A.4), market maker-m chooses w_m to maximize $(u_m + v_m + w_m)(u_m + v_m)$ taking into account condition in (A.9). After simplification, the first order condition becomes

$$w_m(16\lambda_{mn} + 2\lambda_n) = 3W\frac{\lambda_n^2}{\lambda_{mn}}. \tag{A.10}$$

This implies that

$$16\sum_{m=1}^{M}\lambda_{mn}w_m = W\lambda_n.$$

Since $\lambda_{mn} > \lambda_n > 0$, the last equation implies that $w_m = W = 0$. Note that the second order condition for (A.10) is always satisfied. From (A.7) and (A.8) we obtain that $u_m = v_m = 0$ for all m. This concludes the proof of the lemma. □

Proof of Proposition 2. When M approaches infinity, the equilibrium in Proposition 1 can be simplified as follows:

$$\Delta x_n = \beta_n(v^* - v_{n-1})\Delta t_n,$$

$$p_n = p_{mn} = v_{n-1} + \lambda_n(\Delta x_n + \Delta z_n),$$

$$E[\pi_n^I \mid \mathscr{F}_{n-1}, v^*] = a_{n-1}(v^* - v_{n-1})^2 + b_{n-1},$$

$$E[\pi_{mn}^M \mid \mathscr{F}_{n-1}] = 0.$$

Given $\rho_0 = \sigma_z/\sigma_v$, constants $\beta_n, \lambda_n,\ \rho_n,\ a_n$, and b_n are uniquely determined by the following difference equation system

$$\lambda_n = \frac{\beta_n}{\rho_n^2},$$

$$\beta_n\Delta t_n = \frac{1 - 2a_n\lambda_n}{2\lambda_n(1 - a_n\lambda_n)},$$

$$a_{n-1} = \frac{1}{4\lambda_n(1 - a_n\lambda_n)},$$

$$b_{n-1} = b_n + a_n \lambda_n^2 \sigma_z^2 \Delta t_n,$$

$$\rho_{n-1}^2 = \rho_n^2 - \beta_n^2 \Delta t_n,$$

subject to $a_N = b_N = 0$ and the second order condition

$$\lambda_n(1 - a_n\lambda_n) > 0.$$

Since $v_n = v_{n-1} + (\beta_n/\rho_n^2)(\Delta x_n + \Delta z_n) = p_n$, it is easy to see now that the above equilibrium and sequential equilibrium in Kyle (1985) are described by exactly the same system of difference equations and final conditions. (See Kyle (1985, pp. 1322–1323.)) □

Proof of Proposition 3. Part (1) of the proposition follows from the homogeneity of the difference equation in Proposition 1 with respect to σ_v and σ_z. When σ_v changes by factor k_v and σ_z changes by factor k_z, then equilibrium constants λ_n, β_n, ρ_n, a_n, b_n, and c_n change by factors k_v/k_z, k_z/k_v, k_z/k_v, k_z/k_v, $k_v k_z$, and $k_v k_z$, respectively.

To prove Parts (2) and (3) we first define a new variable r_n and derive a difference equation for this variable which provides an explicit method for solving the system of the difference equation in Proposition 1.[25] Then we use the properties of the sequence of r_n to prove the comparative statics with respect to n an M.

Let $s := 1/(M-2) > 0$. Define a new variable

$$r_n := \frac{1 - 2a_n\beta_n/\rho_n^2}{1 + 2s}.$$

From (9) we obtain

$$\frac{a_{n-1}}{a_n} = \frac{1 - 2sr_n}{(1 + r_n)(1 - (1 + 2s)r_n)}. \tag{A.11}$$

Eqs. (7), (8) and (12) can now be rewritten as

$$\frac{\lambda_{n-1}}{\lambda_n} = \frac{1 - (1 + 2s)r_{n-1}}{1 - (1 + 2s)r_n}\frac{a_n}{a_{n-1}} = \frac{(1 + r_n)(1 - (1 + 2s)r_{n-1})}{1 - 2sr_n}, \tag{A.12}$$

$$\frac{\beta_{n-1}}{\beta_n}\frac{\Delta t_{n-1}}{\Delta t_n} = \frac{r_{n-1}}{r_n}\frac{1 - 2sr_n}{(1 + r_{n-1})(1 - (1 + 2s)r_{n-1})}, \tag{A.13}$$

$$\frac{\rho_{n-1}^2}{\rho_n^2} = \frac{1}{(1 + r_n)}. \tag{A.14}$$

[25] The method we use to solve the difference equation in Proposition 1 is similar to recursions used in Holden and Subrahmanyam (1992), and Foster and Viswanathan (1993) for the model with several informed traders and competitive market makers.

From the definition of r_n we also have

$$\frac{\beta_{n-1}}{\beta_n} = \frac{1-(1+2s)r_{n-1}}{1-(1+2s)r_n}\,\frac{a_n}{a_{n-1}}\,\frac{\rho_{n-1}^2}{\rho_n^2} = \frac{1-(1+2s)r_{n-1}}{1-2sr_n}. \tag{A.15}$$

Finally, comparing (A.13) and (A.15) yields the difference equation for r_n

$$\frac{\Delta t_{n-1}}{\Delta t_n}(1-(1+2s)r_{n-1})^2(1+r_{n-1}) = r_{n-1}\frac{(1-2sr_n)^2}{r_n}. \tag{A.16}$$

Difference equation (A.16) expresses r_{n-1} in terms of r_n. Given $0<r_n<1/2s$, this cubic equation always has three real roots, one in $(-\infty,-1)$, one in $(0,1/(1+2s))$, and one in $(1/(1+2s),\infty)$. Of these three roots, only the middle one satisfies the second order condition (13) which requires that $(1+r_n)(1-(1+2s)r_n)>0$. Therefore, using difference equation (A.16), the sequence of r_n can be uniquely iterated backwards starting with boundary condition $r_N = 1/(1+2s)$.

Part (2). Since $r_n>0$ for all $n<N$, it immediately follows from (A.14) that $\rho_n>\rho_{n-1}$ and $\sigma_n<\sigma_{n-1}$.

Now assume that auctions are held at equally spaced intervals: that is, for all n, $\Delta t_{n-1}/\Delta t_n = 1$ in (A.16). To prove relationships $\beta_n>\beta_{n-1}$ and $\lambda_n<\lambda_{n-1}$, we use representations in (A.15) and (A.12). We need to show that for all $r_n\in(0,1/2s)$, the solution of (A.16) r_{n-1} satisfies the following two inequalities:

$$\frac{2s}{1+2s}r_n<r_{n-1}, \tag{A.17}$$

$$r_{n-1}<\frac{r_n}{1+r_n}. \tag{A.18}$$

Define $F(r)=F(r;r_n,s)$ as

$$F(r) := (1-(1+2s)r)^2(1+r) - r\frac{(1-2sr_n)^2}{r_n}.$$

Observe that equation $F(r)=0$ has the unique solution in $(0,1/(1+2s))$ and that $F(0)>0$, $F(1/(1+2s))<0$, and $F(r_{n-1})=0$. Therefore, to prove (A.17) and (A.18) it is sufficient to show that $F((2s/(1+2s))r_n)>0$ and $F(r_n/(1+r_n))<0$, respectively. The first inequality follows from

$$F\left(\frac{2s}{1+2s}r_n\right) = (1-2sr_n)^2\left(1+\frac{2sr_n}{1+2s}\right) - \frac{2sr_n}{1+2s}\,\frac{(1-2sr_n)^2}{r_n} = \frac{(1-2sr_n)^3}{1+2s}>0.$$

The second inequality follows because

$$F\left(\frac{r_n}{1+r_n}\right)\frac{(1+r_n)^3}{(1-2sr_n)^2} = (1+2r_n)-(1+r_n)^2 = -r_n^2<0.$$

To show that δ_n^2 increases over time, we write

$$\delta_n^2 = \mathrm{var}\{p_n \mid \mathscr{F}_{n-1}\} = \lambda_n^2 E[(\Delta x_n + \Delta z_n)^2 \mid \mathscr{F}_{n-1}] = (1+s)^2(\sigma_{n-1}^2 - \sigma_n^2).$$

Therefore,

$$\frac{\delta_{n-1}^2}{\delta_n^2} = \frac{\sigma_{n-2}^2 - \sigma_{n-1}^2}{\sigma_{n-1}^2 - \sigma_n^2} = \frac{r_{n-1}}{r_n}(1 + r_n) < 1,$$

where we used (A.18). The last inequality also implies that σ_n^2 decreases with n at an increasing rate.

Remark. Note that in the limiting case of $M = \infty$ (i.e., $s = 0$) inequality (A.17) becomes simply $0 < r_{n-1}$, which is always satisfied (even when auctions are *not* equally spaced). □

Proof of Proposition 4. Part (2). From Proposition 1, we have

$$v^* - v_n = \frac{\rho_{n-1}^2}{\rho_n^2}(v^* - v_{n-1}) - \frac{\beta_n}{\rho_n^2}\Delta z_n. \tag{A.19}$$

Since Δz_n is serially uncorrelated and independent of v^*, it is straightforward to show that

$$E[(v^* - v_n)(v^* - v_{n-k})] = \sigma_n^2.$$

For all n,

$$v^* - p_n = (v^* - v_{n-1})(1 - \lambda_n \beta_n \Delta t_n) - \lambda_n \Delta z_n.$$

Therefore, using (A.19) obtain that

$$\begin{aligned} &E[(v^* - p_n)(v^* - p_{n-k})] \\ &= (1 - \lambda_n \beta_n \Delta t_n)(1 - \lambda_{n-k}\beta_{n-k}\Delta t_{n-k}) E[(v^* - v_{n-1})(v^* - v_{n-k-1})] \\ &\quad + (1 - \lambda_n \beta_n \Delta t_n)\frac{\beta_{n-k}}{\rho_{n-1}^2}\lambda_{n-k} E[\Delta z_{n-k}^2] = (1 - \lambda_n \beta_n \Delta t_n)\sigma_{n-1}^2 > 0. \end{aligned}$$

The last inequality follows the proof of Proposition 3 because $r_n \in (0, 1/(1 + 2s))$ and

$$1 - \lambda_n \beta_n \Delta t_n = \frac{1 - r_n}{1 + r_n} > 0.$$

Part (3). From Proposition 1, for all n

$$p_{n+1} - p_n = -\frac{1}{M-2}\frac{\beta_n}{\rho_n^2}(\Delta x_n + \Delta z_n) + \lambda_{n+1}(\Delta x_{n+1} + \Delta z_{n+1}).$$

The claim follows because

$$E[(\Delta x_n + \Delta z_n)(\Delta x_{n-k} + \Delta z_{n-k})] = 0 \quad \text{for all } k \geqslant 1,$$

and

$$E[(\Delta x_n + \Delta z_n)^2] = \sigma_{n-1}^2 \rho_n^2 \Delta t_n. \qquad \square$$

Proof of Proposition 5. The proposition follows immediately from comparative statics for λ_n in Proposition 3 and the fact that

$$S_n^{\mathrm{a}} = \frac{M-2}{M-1}\lambda_n, \quad S_n^{\mathrm{c}} = \frac{1}{M-1}\lambda_n. \qquad \square$$

Proof of Proposition 6. Consider a sequence of partitions τ_k such that $\Delta h \to 0$. Assume that functions $\beta^{\tau_k}(t)$, $\lambda^{\tau_k}(t)$, $a^{\tau_k}(t)$, $b^{\tau_k}(t)$, $c^{\tau_k}(t)$, and $\rho^{\tau_k}(t)$ converge to some continuously differentiable functions $\beta(t)$, $\lambda(t)$, $a(t)$, $b(t)$, $c(t)$ and $\rho(t)$ on $[0, 1)$. Then these functions must solve the following system of differential equations (where we suppress dependence on t).

For all $t \in [0, 1)$

$$\lambda = \frac{M-1}{M-2}\frac{\beta}{\rho^2}, \tag{A.20}$$

$$0 = 1 - 2a\frac{\beta}{\rho}, \tag{A.21}$$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = a\left(2\frac{\beta^2}{\rho^2} - \beta\right), \tag{A.22}$$

$$\frac{\mathrm{d}b}{\mathrm{d}t} = -a\sigma_z^2\frac{\beta^2}{\rho^4}, \tag{A.23}$$

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\frac{\sigma_z^2}{M(M-2)}\frac{\beta}{\rho^2}, \tag{A.24}$$

$$\frac{\mathrm{d}\rho^2}{\mathrm{d}t} = \beta^2. \tag{A.25}$$

From (A.21) and (A.22), function $a(t)$ must be constant, $a(t) = A$. Then from (A.21) and (A.25),

$$\frac{\mathrm{d}\rho^2}{\mathrm{d}t} = \frac{\rho^4}{4A^2}.$$

Solving this equation with boundary conditions $\rho(0) = \rho_0$ and $\sigma(1) = \sigma_z/\rho(1) = 0$ (at time $t = 1$, no uncertainty about the asset's liquidation value remains)[26] we find that

$$\rho^2(t) = \frac{\rho_0^2}{1-t}, \quad a = A = \frac{\rho_0}{2}, \quad \beta(t) = \frac{\rho_0}{1-t}.$$

Then from (A.20), (A.23) and (A.24)

$$\lambda(t) = \frac{M-1}{M-2}\frac{1}{\rho_0}, \qquad b(t) = \frac{\sigma_z^2}{2\rho_0}(1-t), \qquad c(t) = \frac{1}{M(M-2)}\frac{\sigma_z^2}{2\rho_0}(1-t).$$

The second order condition (13) is satisfied because it now becomes

$$\frac{M}{2(M-2)}\frac{1}{\rho_0} > 0.$$

From the standard results for converting finite differences into difference equations we obtain that functions $\beta^{\tau_k}(t)$, $\lambda^{\tau_k}(t)$, $a^{\tau_k}(t)$, $b^{\tau_k}(t)$, $c^{\tau_k}(t)$, and $\rho^{\tau_k}(t)$ converge to $\beta(t)$, $\lambda(t)$, $a(t)$, $b(t)$, $c(t)$ and $\rho(t)$, respectively, and that the convergence is uniform for each $[0, t']$ where $t' < 1$. □

References

Admati, A.R., Pfleiderer, P., 1988. A theory of intraday trading patterns. Review of Financial Studies 1, 3–40.

Back, K., 1992. Insider trading in continuous time. Review of Financial Studies 5, 387–409.

Bhattacharya, U., Spiegel, M., 1991. Insiders, outsiders and market breakdowns. Review of Financial Studies 4, 225–282.

Biais, B., Martimort, D., Rochet, J.-C., 2000. Competing mechanisms in a common value environment. Econometrica 68, 799–838.

Bernhardt, D., Hughson, E., 1997. Splitting orders. Review of Financial Studies 10, 69–101.

Black, F., 1991. Trading in equilibrium with bluffing, credits and debits. Working Paper, Goldman Sachs.

Bondarenko, O., Sung, J., 2000. Specialist participation and limit orders. Working Paper, UIC.

Christie, W., Harris, J., Schultz, P., 1994. Why did NASDAQ market makers stop avoiding odd-eighth quotes? Journal of Finance 49, 1841–1860.

Christie, W., Schultz, P., 1994. Why do NASDAQ market makers avoid odd-eighth quotes? Journal of Finance 49, 1813–1840.

Chung, K.H., Charoenwong, C., 1998. Insider trading and the bid-ask spread. Financial Review 33, 1–20.

Dennert, J., 1993. Price competition between market makers. Review of Economic Studies 60, 735–751.

Dutta, P., Madhavan, A., 1997. Competition and collusion in dealer markets. Journal of Finance 52, 245–276.

Easley, D., O'Hara, M., 1987. Price trade size, and information in securities markets. Journal of Financial Economics 19, 69–90.

[26] Note that in the proof we do not use boundary conditions $a_N = b_N = 0$ because some of the equilibrium functions are not defined at $t = 1$. Instead, we use boundary condition $\lim_{t\to 1} \sigma_z/\rho(t) = 0$ which must hold because for any partition τ $\lim_{t\to 1} \sigma_z/\rho^{\tau_k}(t) = 0$.

Easley, D., O'Hara, M., 1992. Time and the process of security price adjustment. Journal of Finance 47, 577–606.

Foster, F.D., Viswanathan, S., 1993. Variations in trading volume, return volatility, and trading costs: evidence on recent price formation models. Journal of Finance 48, 187–211.

George, T., Kaul, G., Nimalendran, M., 1991. Estimation of the bid–ask spread and its components. Review of Financial Studies 4, 623–656.

Glosten, L., 1994. Is the electronic open limit order book inevitable? Journal of Finance 49, 1127–1161.

Glosten, L., 1989. Insider trading, liquidity, and the role of the monopolist specialist. Journal of Business 62, 211–235.

Glosten, L., Harris, L., 1988. Estimating the components of the bid/ask spread. Journal of Financial Economics 21, 123–142.

Glosten, L., Milgrom, P., 1985. Bid, ask, and transaction prices in a specialist market with heterogeneously informed traders. Journal of Financial Economics 13, 71–100.

Hasbrouck, J., Sofianos, G., 1993. The trades of market makers: an empirical analysis of NYSE specialists. Journal of Finance 48, 1565–1593.

Holden, C.W., Subrahmanyam, A., 1992. Long-lived private information and imperfect competition. Journal of Finance 47, 247–270.

Huang, R., Stoll, H., 1997. The components of the bid–ask spread: a general approach. Review of Financial Studies 10, 995–1034.

Huang, R., Masulis, R., 1999. FX spreads and dealer competition across the 24 hour trading day. Review of Financial Studies 12, 61–93.

Kavajecz, K.A., 1998. A specialist's quoted depth as a strategic choice variable. Working Paper, Wharton School, University of Pennsylvania.

Kyle, A., 1984. Market structure, information, futures markets, and price formation.. In: Storey, G., Schmitz, A., Sarris, A. (Eds.), International Agricultural Trade: Advanced Readings in Price Formation, Market Structure, and Price Instability.. Westview Press, Boulder, Co, pp. 45–64.

Kyle, A., 1985. Continuous auctions and insider trading. Econometrica 47, 331–359.

Kyle, A., 1989. Informed speculation with imperfect competition. The Review of Economic Studies 56, 317–356.

Madhavan, A., 1992. Trading mechanisms in securities markets. Journal of Finance 47, 607–642.

Madhavan, A., Richardson, M., Roomans, M., 1997. Why do security prices change? A transaction-level analysis of NYSE stocks. Review of Financial Studies 10, 1035–1064.

McInish, T., Wood, R., 1992. An analysis of intraday patterns in bid/ask spreads for NYSE stocks. Journal of Finance 47, 753–764.

O'Hara, M., Oldfield, G., 1986. The microeconomics of market making. Journal of Financial and Quantitative Analysis 21, 361–376.

Seppi, D., 1997. Liquidity provision with limit orders and a strategic specialist. Review of Financial Studies 10, 103–150.

Sofianos, G., 1995. Specialist gross trading revenues at the New York Stock Exchange. NYSE Working Paper #95-01.

Stoll, H., 1989. Inferring the components of the bid–ask spread: theory and empirical tests. Journal of Finance 44, 115–134.

Subrahmanyam, A., 1991. Risk aversion, market liquidity, and price efficiency. Review of Financial Studies 4, 417–442.