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Specialist participation and limit orders ☆

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Abstract

We present a market microstructure model to examine specialist's strategic participation decisions in a security market where there are noise traders, limit order traders, an insider and a specialist. We argue that the specialist's participation rate depends on the depth of the limit book and its uncertainty. In particular, the specialist has incentives to trade against the market trend when the limit book depth is low and to trade with the market trend when the depth is high. Moreover, the specialist's participation rate is positively related to the limit book depth uncertainty and the asset price volatility, but is negative related to the average trading volume. We also discuss the specialist's participation strategies under the NYSE regulation that prohibits the specialist from trading with the market trend.

JEL classificaion: D42; D82

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1. Introduction

Many financial markets rely on market makers, or specialists, for the provision of liquidity. The trading behavior of market makers plays important role in price formation and, therefore, it is of considerable interest to traders, regulators, and researches.

Recently, there has been a rapidly growing literature that investigates optimal trading behavior of one or more market makers under different information

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environments. Glosten (1989) examines specialist's strategic price-setting behavior against market orders from informed and uninformed traders. Kavajecz (1998) extends Glosten by allowing the specialist to choose quantities of trades as well as their prices. Dennert (1993) and Bernhardt and Hughson (1997) analyze price competitions among several market makers, when informed and uninformed traders are allowed to split market orders among market makers.[1] All of the above, however, assume that the market maker/specialist provides liquidity for all incoming market orders, i.e., that the specialist always participates in security transactions.

Many real exchanges, however, operate as *hybrid* markets, where liquidity is supplied by both specialists and public limit orders. On the NYSE and Amex, for example, designated specialists must reflect in their quotes the highest bid and the lowest ask prices in the limit order book when these limit prices are better than their own quotes. In hybrid markets, the reliance on the specialist to provide liquidity may vary considerably across stocks. Empirical study by Madhavan and Sofianos (1998), for example, reports that some stocks on the NYSE essentially trade in quote-driven dealer markets, where up to 90% of liquidity is provided by the specialist, while some other stocks trade in essentially order-driven auction markets, where the specialist participates in less than 10% of trades. In another empirical study, Harris and Hasbrouck (1996) find that limit orders represent about 54% of all orders submitted through SuperDOT. These findings suggest that the interaction between the specialist and the limit order book plays important role in price formation and that the theoretical models in which liquidity is supplied by specialist only or by limit order traders only[2] might not be suitable for predicting price-setting behavior in many hybrid markets.

The objective of this paper is to develop an analytical framework of the specialist's trading behavior in the presence of direct competition from limit order traders. We study a call market where the specialist and limit order traders provide liquidity to incoming market orders from noise traders and an insider. We extend Kyle's (1985) market structure to allow the specialist to execute market orders submitted by the insider and noise traders, using public limit orders as well as his own inventory.[3] In our model, the depth of the limit book is a random variable, reflecting the fact that some limit orders may come from noise limit traders. The realized depth of the limit book is observable to the specialist, but unobservable to other traders. Since only he can observe the state of the limit order book, the specialist has an important informational advantage over the public. When setting a market clearing price, the specialist optimally chooses whether and

[1] See also Biais et al. (2000) and Bondarenko (2001).

[2] For pure limit order book systems (without a specialist), see Glosten (1994), and Biais et al. (1995).

[3] In Kyle (1985), there are one risk-neutral insider, liquidity traders and a market maker. Later, his model was extended to accommodate richer market structures. See Holden and Subrahmanyam (1992) for a model with multiple informed traders. For a continuous-time generalization, see Back (1992). Subrahmanyam (1991) extends the basic model to allow for a risk-averse insider. Strategic behavior of liquidity traders is studied in Admati and Pfleiderer (1988).

how much to trade from his own inventory. Hence, his participation decisions arise nontrivially.[4]

Our paper is similar in spirit to Seppi (1997), who also looks at the interaction of a specialist and limit order traders. In Seppi, the competitive limit order book and the specialist's optimal pricing rule are derived given an exogenous process for arriving market orders. In our paper, in contrast, the insider's trading strategies (as well as the specialist's pricing rule) are derived endogenously. Furthermore, our focus is on the specialist's participation decisions under different regulatory constraints on specialist dealings.

Other related papers are Kumar and Seppi (1994) and Chakravarty and Holden (1995), which propose models of hybrid markets with limit orders and dealers. The first model is based on an order-driven trading mechanism while the second one employs a quote-driven trading mechanism similar to one in Glosten and Milgrom (1985). In both papers, however, the dealers are competitive and strategic aspects of their behavior in the presence of competition from limit order traders are not addressed.

In this paper, we try to get an insight into the following primary questions: (1) What are the main characteristics of specialist's optimal participation strategies? (2) Under what conditions does the specialist have incentives to participate for his own account against or with the market trend? (3) How does the regulation that prohibits the specialist's trading with the market trend affect the market liquidity and other market outcomes? (4) What is the relationship between the specialist's participation and major microstructure variables such as price volatility and trading volume?

An important special case of our model is when the limit book is *certain* (i.e., nonrandom). A certain limit book may arise when there are no noise limit order traders or when the limit book is completely transparent to the public. When the limit book is certain, the specialist does not have an informational advantage over the public. In this case, the depth of the limit book is determined by perfect competition among limit order traders, who collectively drive the expected profit on price-contingent limit orders to zero. Since limit order traders compete away any liquidity provision profits, the specialist has no incentives to participate and all liquidity is provided by limit orders. It is not surprising, therefore, that our certain limit book system completely replaces the role of the specialist in Kyle (1985), resulting in exactly the same market outcomes.

An important question then arises as to whether the specialist would ever have incentives to participate in security dealings. We argue that, if the limit book is *uncertain*, then the specialist indeed has incentives to participate. We show that the strategic specialist participates against (with) the market trend when the realized depth of the limit order book is less (greater) than some critical value, the particular value of the limit book depth conditional on which the expected profit to limit orders is zero.

[4] In our model, the specialist is risk-neutral and the inventory control costs play no role. For references on inventory models, see Hasbrouck and Sofianos (1993) and Madhavan and Smidt (1993).

When the realized depth is lower than the critical value, the potential profit opportunities from liquidity provision are not fully exploited. Hence, the specialist has incentives to join limit orders and to trade *against* the market trend by providing more liquidity to the market. When the realized depth is higher than the critical value, supplying additional liquidity is no longer profitable; instead, the specialist has incentives to consume liquidity by joining market orders, i.e., by trading *with* the market trend and against limit orders.

It is important to note, however, that the NYSE restricts the specialist dealings with the market trend. Specifically, NYSE Rule 104—"Dealings by Specialists"—prohibits the specialist, when he trades for his own account, from: (1) buying at a price above the last transaction price, (2) selling at a price below the last transaction price, (3) buying nearly all of the standing limit sell orders or more than 50% of an arriving limit sell order at a price equal to the last transaction price, and (4) selling nearly all of the standing limit buy orders or more than 50% of an arriving limit buy order at a price equal to the last transaction price. The purpose of this regulation is to prevent the specialist from consuming liquidity in the market. In our model, this regulation implies that the specialist is prohibited to trade with the market trend. Therefore, to be consistent with the current practices of the NYSE, we investigate the effects of such a regulation on the specialist's trading strategies.

Whether he is restricted or not, the specialist's *participation rate*, the ratio of the specialist's share purchases and sales to total share volume of all market orders, increases with the uncertainty of the limit book depth. Intuitively, as the depth uncertainty increases, the specialist's informational advantage about the state of the limit book increases, which in turn undermines the value of the private information to the insider, making the insider to trade less aggressively. As the insider becomes more conservative, the market has less severe adverse-selection problem and the specialist's participation rate increases.

Intuitively, in a quote-driven trading system, Copeland and Galai (1983) view the market maker's bid and ask quotes as the out-of-the-money strike prices of a put and a call options offered to informed traders. As a similar analogy, in our hybrid market, it is the specialist that is given an option to opportunistically add or remove liquidity to or from the market. As the uncertainty of the limit book increases, the option value also increases.

Our model sheds some light on the following important questions that have been addressed in the empirical literature. How does the specialist's participation vary across stocks? Specifically, what is the relation between the specialist's trading and major microstructure variables such as price volatility and trading volume? We show that, regardless of the regulation, the specialist's participation rate is positively related to the asset price volatility and is negatively related to the trading volume. These relationships arise because the specialist's participation rate and price volatility increase while the trading volume decreases with the uncertainty of the limit book depth. These theoretical relationships are supported by the recent empirical findings in Madhavan and Sofianos (1998), Sofianos (1995) and Chung et al. (1999).

As noted above, the specialist's participation rate critically depends on the depth of the limit order book. Then the next question is how the depth is determined. We show that the depth increases either as the asset value volatility decreases or as the noise trading increases. To see this intuitively, note that, in an equilibrium, limit order traders face a tradeoff between losses to the insider (and the specialist when he is allowed to trade with the market trend) and the profits from trading with noise traders. Therefore, when the uncertainty of the asset value increases or when the noise trading decreases, the adverse-selection problem becomes more severe and limit traders decrease their order sizes.[5]

Finally, our analytical framework allows us to compare relative merits of the two market organization, with and without the NYSE regulation that prohibits the specialist to trade with the market trend. Our results confirm the intuition that the market order traders (noise traders and the insider) are better off under the regulation. In particular, desirable properties of the regulation are that the costs of trading via market orders decrease, noise traders' welfare improves, and the security prices are more informative. On the other hand, of the two market organizations, the specialist prefers the market without the regulation.

Another important policy implication of our analysis is that, regardless of the regulation, the market order traders benefit from making the limit book as certain as possible. This could be achieved by making the book more transparent to the public. In the limiting case of the completely certain limit book, the trading costs are the lowest, noise traders and the insider welfares are the highest, and the security prices are the most informative. In contrast, the specialist prefers when the uncertainty of the limit book depth is high. Intuitively, making the limit book less uncertain removes the informational advantage of the specialist and reduces the costs of trading via market orders.

The remainder of the paper is organized as follows. The next section presents the basic model that applies to all cases to be discussed in this paper. In Section 3, we consider the special case of the model when the limit book is certain. Here we find that in an equilibrium, the specialist does not trade for his own account. In Section 4, we consider a market organization in which the limit book is uncertain and show that non-participation of the specialist is no longer optimal. In Section 4.1, we investigate the specialist's participation strategies when he is allowed to trade freely for his own account with or against the market trend. In Section 4.2, the specialist is not allowed to trade with the market trend. In Section 5, we compare equilibrium outcomes in two market organizations, with and without the restriction on specialist dealings. Section 6 discusses empirical implications of our model. Section 7 provides a brief summary. Proofs of all our results are collected in the appendix.

[5]As a different explanation of the dynamics of the limit book, Rock (1996) argues that the depth increases as the inventory of a risk-averse specialist increases. Intuitively, as the specialist's holding of stock increases, the increased inventory risk causes him to reduce trading, which in turn deepens the book of limit orders.

2. The basic model

There is a single period with two dates, 0 and 1. An asset is traded in a security market where there will be a public release of information at date 1 about the value of the asset. The date-1 value of the asset v is a normal random variable with mean v_0 and variance $\sigma_v^2 > 0$. The market for the asset consists of four types of risk-neutral participants: an insider who, at time 0, already knows v; a specialist with market-making power; noise traders who have random and price-inelastic demand for the traded asset; and a number of traders who compete using limit orders.

The insider and noise traders are active traders on the floor of the exchange who demand liquidity. They simultaneously submit market orders denoted by x and z shares.[6] Noise traders trade for some unspecified exogenous reason. Their cumulative market order, z, is a normal random variable with mean zero and variance $\sigma_z^2 > 0$, and it is independent of v. When submitting her market order, the insider does not observe z. Throughout the paper, we use a convention for all market participants that positive orders represent buy orders and negative orders represent sell orders. Uncertainties about v and z jointly create a trading environment with a nontrivial adverse-selection problem, which depends on comparative sizes of σ_v and σ_z and is captured by the following statistic:

$$\rho := \frac{\sigma_z}{\sigma_v}.$$

Intuitively, ρ is a measure of the market noise relative to the asset value uncertainty and is inversely related to the degree of adverse-selection problem. We refer to ρ as the "noise-to-asset volatility ratio".

Another type of public traders are the limit order traders. They supply liquidity to the market by posting limit-buy order for prices below v_0 and limit-sell orders for prices above v_0. To model the limit book, we assume that limit orders can be placed for all prices and that the limit order book has a uniform depth $\alpha > 0$, measured in shares per dollar. This means that cumulative limit order supply schedule is linear and that, given α, the total volume of limit orders with limit prices up to p is $\alpha(v_0 - p)$. Positive (negative) quantity $\alpha(v_0 - p)$ represents limit-buy (limit-sell) orders.[7]

[6] For tractability, we assume that the insider only uses market orders but not limit orders. This is perhaps because (1) the insider is concerned about revealing her private information to the specialist if she places limit orders or (2) the insider's information is short-lived and thus she seeks the immediacy of a market order before the private information becomes public. For a case where the insider uses both types of orders, see Kumar and Seppi (1994) and Chakravarty and Holden (1995). They do not, however, consider the specialist's participation decisions.

[7] In modeling the limit order book, we abstract from a number of important features of real markets. In reality, for example, limit orders can only be placed on a grid of prices defined by the minimum tick size. This discreteness has been shown to have important effects on the nature of the competition between the specialist and limit order traders. See, for example, Harris (1991) and Seppi (1997). Another examples are the time priority rule, according to which limit orders placed at the same price must be differentiated by time, and the public priority rule, according to which the public limit orders have precedence over orders that the specialist may want to execute on his own account.

The depth of the limit book α is modeled as a random variable with distribution $f(\alpha)$ known to all market participants.[8] In equilibrium, distribution $f(\alpha)$ satisfies the condition that limit order traders earn zero expected profit. One interpretation of our limit book is that some limit orders come from noise limit order traders (hence, the randomness of the limit book), but some come from discretionary traders who ensure that all positive opportunities from liquidity provision are exploited on average.[9]

All public traders, namely, the insider, noise traders and limit book traders, submit their orders to the specialist. The specialist observes the depth of the limit order book α and the total market order $d = x + z$ but cannot distinguish between informed trades x and uninformed trades z. The role of the specialist is to set a single price that clears the market. In doing so, he not only crosses public buy and sell orders, but trades for his own account. Let y denote the order size executed by the specialist for his own account. When the specialist chooses y, he effectively determines the market clearing price p at which orders of all participants are filled. Specifically, the market clearing condition is

$$x + y + z + \alpha(v_0 - p) = 0.$$

The above condition states that all orders are executed through a *nondiscriminatory* auction. This means that if, for example, a market order to buy d shares arrives, then the first limit sell orders totaling d shares all transact at the same price p (equal to the limit price of the highest executed sell order).[10] Note that, when y and d are of the opposite signs, i.e., $yd < 0$, the specialist trades *against* the market trend, providing more liquidity to the market. When $yd > 0$, the specialist trades *with* the market trend, in a sense consuming liquidity.

To examine strategic interactions among traders and public traders, we consider the Perfect Bayesian equilibria. In these equilibria:

(1) the insider chooses a market order x to maximize her expected profit conditional on observed v,

[8] The assumption that the limit book has a uniform depth is a restrictive one. However, it provides a convenient way to characterize the state of the limit book with a single parameter, making the analysis tractable. The assumption could be viewed as a reasonable description of a reality in a continuous-time context, because in a very small time interval, the asset price would not change much, and an arbitrary shape of the book in the neighborhood of $p = v_0$ can be approximated by a linear function of $(p - v_0)$. The assumption of linear limit order supply schedules is also used in a number of important microstructure models, including Kyle (1989), Black (1995), and Glosten (1994).

[9] In an alternative interpretation, the randomness of the depth of the limit book could occur if the number of discretionary limit order traders is random. This approach is taken in Baruch (1997), where a random number of limit order traders submit the same linear demand schedule. As a result, the depth of the limit order book is uniform (because of linearity of demand schedules) and random (because of the random number of the limit order traders).

[10] This type of an auction is used by the NYSE during the preopening call auction and after a trading halt. The nondiscriminatory (or batch) auction assumption, also used in the Kyle (1985, 1989) models, provides a convenient way to capture price impacts resulting from market-order flows in a single-period model. See also Glosten (1994) for a discussion of this assumption.

(2) the specialist chooses y (and thus the market clearing price p) to maximize his expected profit conditional on observed limit book depth α and the market order d, and
(3) the expected profit of limit traders is zero.

Formally,

Definition 1. A triplet $(x(v), p(\alpha, d), f(\alpha))$ is called a security market equilibrium if the following conditions are satisfied.

(1) Insider's profit maximization: $x(v) \in \arg\max_{x'} \mathrm{E}[(v - p(\alpha, d))x' \mid v]$.
(2) Specialist's profit maximization: $p(\alpha, d) \in \arg\max_{p'} \mathrm{E}[(v - p')(\alpha(p' - v_0) - d) \mid \alpha, d]$.
(3) Perfect limit order competition: for $f(\alpha)$, $\mathrm{E}[(v - p(\alpha, d))\alpha(v_0 - p(\alpha, d))] = 0$.

Definition 1, however, is too general to be tractable. Therefore, we focus on the popular class of pricing rules which are linear in market order d, that is

$$p(\alpha, d) = \delta(\alpha) + \lambda(\alpha)d. \quad (1)$$

We refer to a security market equilibrium with linear pricing rule in (1) as a *linear market equilibrium*. As will be seen later on, linear pricing rule in (1) implies that the insider's strategy is also a linear function $x(v) = \gamma + \beta v$, for constant γ and β.[11]

3. Certain limit book

We study the main case of the model in the next section. Before that, however, we develop intuition by considering the special case when the limit book is *certain*, i.e, nonrandom. In this case, the depth of the limit book α is a constant known to all market participants. A certain limit book may arise if either there are no noise limit order traders or the limit book is completely transparent to the public so that at any moment there are no profit opportunities left in the book. We refer to this special case as the CB (certain book) market.

In the CB market, there are only two types of uncertainty: noise trading volume and the asset's future value. The insider who faces only the first type of uncertainty attempts to extract the maximum possible rent from her private information. It is also intuitively clear that, since the specialist does not have an informational advantage over the public, he does not have incentives to trade from his own inventory. This is because all profit-making opportunities are competed away by limit order traders. This intuition is confirmed by the following proposition.

[11] Several papers investigate equilibria when pricing schedules are allowed to be *nonlinear*, starting with Bhattacharya and Spiegel (1991).

Proposition 1. *The CB market is in a linear equilibrium if and only if the following three conditions are satisfied*:

(1) *the insider's trading strategy is* $x(v) = \beta(v - v_0)$, *where* $\beta = \rho$;
(2) *the specialist's pricing rule is* $p(d) = v_0 + \lambda d$, *where* $\lambda = 1/(2\rho)$;
(3) *the depth of the limit book is* $\alpha = 2\rho$.

Proposition 1 implies that the specialist's participation is always zero, because

$$y(d) = -\alpha(v_0 - p(d)) - d \equiv 0.$$

In other words, when the book is certain, all liquidity is exclusively supplied by limit order traders. The pricing rule in the CB market is determined by perfect competition of limit order traders and it is the same as the pricing rule of Kyle (1985), meaning that our certain limit order book completely replaces the role of the specialist.

Note that in our model the specialist's non-participation in the CB market is driven by two assumptions, that (1) all limit orders are executed at the same "clean-up" price rather than at the posted limit prices, and (2) limit order traders break-even on average. If limit orders were executed at their posted prices and/or if the limit order traders had to defray some up-front order commission costs (as, for example, in Seppi, 1997), then the limit book would be less deep and there would be room for the specialist to participate even with a certain book.

An important question then arises: under what conditions will the specialist have incentives to participate in trading? In the following section, we argue that such incentives exist when the limit book is uncertain.

4. Uncertain limit book

In this section, we assume that the depth of the limit book α is a random variable, independent of v and z. The realization of α is observable to the specialist but unobservable to other market participants. There are two practical reasons for modeling the limit book with random depth parameter. On many organized exchanges (including the NYSE), the trading public does not have full access to the limit book. In other exchanges, the limit order book is officially open but it changes too frequently for traders to keep up, and thus traders may view the depth of the book as random. Moreover, Biais et al. (1995) report that even in a pure open limit order system like the Paris Bourse, there are inevitably hidden orders in the limit books, and the actual depths of the books are not completely known to the traders.[12]

[12] A trader of the Paris Bourse has an option to place a limit order such that only a fraction of the order is visible to other traders, while the remaining fraction is present in the book, but invisible to other traders. Once the visible fraction of the order has been fully executed, another portion of the hidden order (equal to the amount initially disclosed) becomes visible.

When the limit book is uncertain, both the insider and the specialist have private information. The insider knows the asset's future value, but the specialist knows the realized state of the limit book. The informational advantage of the specialist allows him to earn positive profits from making the market. In particular, when the realized depth of the limit book is too low, the monopolistic specialist charges more for liquidity, that is, he clears market buy (sell) orders at higher (lower) prices. When the limit book is very deep, the specialist may find it profitable to trade *with* the market trend, in a sense, consuming the liquidity. Although the NYSE does not allow the specialist to trade with the market trend, we first examine the specialist's trading strategies without imposing such a restriction, and then, in Section 4.2, we investigate implications of the regulation prohibiting the specialist from trading with the market trend.

4.1. Unrestricted specialist dealings

In this subsection, the specialist is allowed to trade both with and against the market trend to maximize his expected profit. We refer to this market organization as the UD (unrestricted specialist dealings) market. To characterize market equilibria, we will need to impose some structure on possible distributions of α. Specifically, for distribution $f(\alpha)$, we define the following statistic:

$$s := \left(\mathrm{E}[\alpha]\mathrm{E}\left[\frac{1}{\alpha}\right]\right)^{1/4}. \tag{2}$$

This *dimensionless* statistic plays an important role in our analysis. Intuitively, it is a measure of dispersion of the limit book depth. This statistics quantifies the uncertainty/randomness of the limit book: $s = 1$, when the limit book is certain, and increases to infinity as the limit book uncertainty increases. High values of s imply that there are large number of noise limit traders in the market. We assume that uncertainty of the limit book s in exogenously given.[13]

We also assume that distribution $f(\alpha)$ satisfies two assumptions stated in the appendix. The first assumption requires existence of certain moments of α, which are needed to compute various equilibrium outcomes. The second one requires that distribution $f(\alpha)$ can be uniquely parameterized by its mean $\mathrm{E}[\alpha]$ and statistic s. While this assumption is not required for deriving the equilibrium conditions, it guarantees that in the following Proposition 2 (Proposition 5) the equilibrium always exists and is unique. As discussed in the appendix, the two assumptions are satisfied by many popular families of distributions, including lognormal, Gamma, and two-point distributions. Later on, to analyze several complex issues, we will use the two-point distributions.

Now we can state the main result of this subsection.

[13] An alternative representation for s follows from equality $s^4 = 1 - \mathrm{cov}(\alpha, 1/\alpha)$, which implies that s increases when distribution of α becomes more uncertain or "spread out".

Proposition 2 (Unrestricted specialist dealings). *In the UD market, there exists unique linear equilibrium. In the equilibrium,*

(1) *the insider's trading strategy is* $x(v) = \beta(v - v_0)$, *where*

$$\beta = \frac{\rho}{s};$$

(2) *the specialist's pricing rule is* $p(\alpha, d) = v_0 + \lambda(\alpha)d$, *where*

$$\lambda(\alpha) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\rho}\frac{s}{s^2+1}\right);$$

(3) *the limit book depth* α *is distributed such that* $\mathrm{E}[\alpha] = \rho(s + s^3)$.

Proposition 2 characterizes the market equilibrium $(x(v), p(\alpha, d), f(\alpha))$ in terms of exogenous parameters, noise-to-asset-volatility ratio ρ and the uncertainty of the limit book s. In particular, given these parameters, the competition among limit traders forces distribution $f(\alpha)$ to adjust in such a way that its mean $\mathrm{E}[\alpha]$ is equal to $\rho(s + s^3)$. Note also that in the special case when there is no uncertainty in the limit book ($s = 1$), the equilibrium in Proposition 2 reduces to the CB market equilibrium in Proposition 1. The next proposition looks at the comparative statics of the linear equilibrium with respect to ρ and s. Our main interests lie in β (the insider's sensitivity to v), $\mathrm{E}[\alpha]$ (the expected limit book depth), and $\mathrm{E}[\lambda(\alpha)]$ (the expected sensitivity of the pricing rule to the market order flow).

Proposition 3 (Comparative statics for the UD market). *Along the equilibrium path,*

$$\frac{\partial\beta}{\partial\rho} > 0, \quad \frac{\partial\mathrm{E}[\alpha]}{\partial\rho} > 0, \quad \frac{\partial\mathrm{E}[\lambda(\alpha)]}{\partial\rho} < 0$$

and

$$\frac{\partial\beta}{\partial s} < 0, \quad \frac{\partial\mathrm{E}[\alpha]}{\partial s} > 0, \quad \frac{\partial\mathrm{E}[\lambda(\alpha)]}{\partial s} > 0.$$

The intuition for Proposition 3 is as follows. When ρ increases, the insider finds it easier to hide her orders in noise market orders. Hence, she becomes more aggressive to capitalize on the private information, resulting in higher β. Meanwhile, the expected sensitivity of the pricing rule decreases with ρ. These results are consistent with Kyle (1985).

Proposition 3 also tells us that the expected depth of the limit order book increases with the market noise trading: a high level of noise trading attracts more limit orders, thereby increasing the average depth of the limit order book.[14]

[14] To be more precise, an increase in ρ has two opposite effects on the expected depth of the limit order book: an increase in the expected *profits* due to a larger noise trading, and an increase in the expected *losses* due to a more aggressive strategy of the insider. It turns out that the first effect dominates, resulting in a net increase in the depth of the limit book.

On the other hand, as the book depth becomes more uncertain (as s increases), the information asymmetry about the depth grows in favor of the specialist over the insider. Thus, the insider becomes less aggressive to trade on the private information, resulting in a smaller β. As the insider becomes less aggressive, the adverse-selection problem is mitigated, which in turn attracts more limit orders.

We now turn to our main interest, the specialist's participation. By Proposition 2, the specialist's trading strategy is $y(\alpha, d) = (\alpha\lambda(\alpha) - 1)d$. When the book is uncertain, α is no longer a constant but a random variable. In this case, the specialist trading strategy depends on a particular realization of α as follows:

(1) when $\alpha\lambda(\alpha) - 1 < 0$, he trades against the market trend;
(2) when $\alpha\lambda(\alpha) - 1 = 0$, he does not participate in the market; and
(3) when $\alpha\lambda(\alpha) - 1 > 0$, he trades with the market trend.

Further details of the specialist's participation strategies are revealed by considering the (*signed*) *participation rate*, defined as the ratio of the specialist share purchases (sales) to total share volume of all market orders:

$$R(\alpha, d) := \frac{y(\alpha, d)}{d} \quad \text{for all } d \neq 0 \tag{3}$$

and, by convention, we set $R(\alpha, d) = 0$, for $d = 0$.[15] By Proposition 2,

$$R(\alpha, d) = \frac{1}{2\theta}(\alpha - \theta),$$

where

$$\theta := \frac{\rho^2 + \beta^2}{\beta} = \frac{\rho(1 + s^2)}{s}.$$

The above equations imply that the specialist's optimal strategy is not to participate if $\alpha = \theta$; to trade against the market trend if $\alpha < \theta$, and with the market trend if $\alpha > \theta$. In other words, when the depth of the limit book is lower than the critical value θ, the specialist trades for his own account by providing more liquidity to the market; and when the depth is higher than the critical value, he does so by consuming liquidity.

The critical value θ has the following intuitive interpretation. Consider the expected profit to limit order traders conditional on realized depth being α:

$$\mathrm{E}[(v - p(\alpha, d))\alpha(v_0 - p(\alpha, d)) \mid \alpha] = \tfrac{1}{2}\sigma_v^2 \beta\lambda(\alpha)(\theta - \alpha).$$

It follows from this equation, that when $\alpha < \theta$, there are profit opportunities that are not fully exploited by limit orders. Therefore, the specialist capitalizes on these under-exploited opportunities by supplying more liquidity to the market. On the

[15] Note that the definition of the specialist's participation rate in (3) differs from the definition used by the NYSE. The NYSE definition is based on the *total* reported share volume in a stock, which in our model is given by $(|x| + |y| + |z| + |\alpha(p - v_0)|)/2$. In contrast, the participation rate in (3) is based on the market order imbalance $d = x + z = -y - \alpha(v_0 - p)$, which is the sum of the trade volumes from the specialist's participation and executed limit orders. We use d instead of the total volume, because our specialist's participation decision is a function of d.

other hand, when $\alpha > \theta$, the realized depth of the limit order book is too high to be profitable and the specialist finds it profitable to follow the market trend.

The expected participation rate of the specialist is

$$\mathrm{E}[R] = \mathrm{E}\left[\frac{1}{2}\left(\frac{\alpha}{\theta} - 1\right)\right] = \frac{1}{2}(s^2 - 1),$$

which implies that the expected participation rate monotonically increases with the uncertainty of the limit book s. Since $s > 1$ with an uncertain limit book, E[R0. The expected participation rate, however, is not straightforward to interpret, because the quantity R is signed in such a way that it is positive (negative) if the specialist participates with (against) the market trend. Thus, we focus on the following types of the specialist's participation rates:

$$\bar{R}_- := \mathrm{E}[-R\chi_\theta], \quad \bar{R}_+ := \mathrm{E}[R(1 - \chi_\theta)],$$

where χ_θ is an indicator function equal 1 for $\alpha \leqslant \theta$, and zero otherwise. The first of the two quantities is the expected participation rate against market trend, and the other is the expected participation rate with the market trend. Of course, the sum of the two quantities is simply $\mathrm{E}[|R|]$, while their difference $\bar{R}_+ - \bar{R}_-$ is $\mathrm{E}[R]$.

For a clearer understanding of the specialist's participation rate, let us assume that distribution of α belongs to a family of two-point distributions with two parameters $c > 0$ and $a \geqslant 1$. Given these parameters, the depth of the limit book α can take on two values, a "high" value of ca and a "low" value of c/a, with equal probabilities:

$$\mathrm{Prob}(\alpha = ca) = \mathrm{Prob}\left(\alpha = \frac{c}{a}\right) = \frac{1}{2}.$$

Note that parameter a is qualitatively similar to the previously defined statistic s, since they are positively related to each other:

$$s = \left(\mathrm{E}[\alpha]\mathrm{E}\left[\frac{1}{\alpha}\right]\right)^{1/4} = \sqrt{\frac{1}{2}\left(a + \frac{1}{a}\right)}. \tag{4}$$

Hence, a can be interpreted as another measure of the uncertainty of the limit book. In particular, when $a = 1$, the book is certain; when a increases so does the uncertainty of the limit book. As before, parameter a is assumed to be exogenous while parameter c is determined through the perfect competition of limit traders.

By Proposition 2, there is unique linear equilibrium for the class of two-point distributions. It is characterized by

$$\beta = \frac{\rho}{s}, \mathrm{E}[\alpha] = \rho s(1 + s^2), \qquad \mathrm{E}[\lambda(\alpha)] = \frac{s}{2\rho}, \qquad c = \rho\left(s + \frac{1}{s}\right).$$

The expected specialist's participation rates can now be computed explicitly as

$$\bar{R}_- = \frac{1}{2}\left(1 - \frac{1}{a}\right), \quad \bar{R}_+ = \frac{1}{2}(a - 1). \tag{5}$$

The expected participation rates are zero for the certain limit book ($a = 1$), but they are positive when the book is uncertain ($a > 1$). Moreover, the more uncertain the depth of the limit book, the higher the specialist's participation is, both with and against the market trend. Intuitively, higher uncertainty in the book has two effects: (1) it increases the informational advantage of the specialist, and (2) it makes the insider less aggressive, mitigating the adverse-selection problem faced by the specialist. Both effects contribute to more aggressive trading by the specialist.

We now relate the specialist's participation rates to two important microstructure variables, price volatility and trading volume. This will allow us to compare the predictions of our model with recent empirical studies of specialist's trading. As a measure of trading volume in a stock we use $\mathrm{E}[|d|]$, the expected total share volume of all market orders. This definition is slightly different from the definition of the NYSE, which also includes transactions crossed among traders on the floor (see also footnote 15).

The following proposition states that the two participation rates are positively related to the price volatility, and negatively related to the trading volume.

Proposition 4. *In the UD market equilibria under the two-point distribution assumption,*

$$\frac{\partial \mathrm{var}(p)}{\partial \bar{R}_-} > 0, \quad \frac{\partial \mathrm{var}(p)}{\partial \bar{R}_+} > 0, \tag{6}$$

$$\frac{\partial \mathrm{E}[|d|]}{\partial \bar{R}_-} < 0, \quad \frac{\partial \mathrm{E}[|d|]}{\partial \bar{R}_+} < 0. \tag{7}$$

Relationships in Proposition 4 obtain because, as the uncertainty of the limit book depth increases, the specialist's participation rate and the price volatility both increase while the trading volume decreases.[16] A caution is in order, however. Relationships in (6) and (7) are established along the market equilibrium path. That is, two endogenous variables are positively or negatively related to each other across market equilibria. Therefore, the above relationships should not be interpreted as causal relationships as if the specialist's high participation were causing an increase in price volatility and a decrease in trading volume.

Theoretical results in (6) and (7) are supported by the empirical findings in Madhavan and Sofianos (1998), Sofianos (1995) and Chung et al. (1999). They will be discussed in more detail in Section 6.

In the next subsection, we assume that the specialist is prohibited from dealing with the market trend. This means that the specialist is only allowed to add liquidity to the market. This regulation successfully captures the positive aspect of the specialist system (i.e., supplying liquidity to the market when the limit book depth is too low) but removes the negative aspect of the specialist system (i.e., consuming liquidity when the limit book depth is too high). We then show that the same results

[16] Leach and Madhavan (1993) also show that the negative relationship exists between specialist's participation and trading volume. They assume that a low trading volume implies low competition among traders, and argue that since the low competition gives more freedom for the specialist to more actively experiment with bid/ask prices for price discovery, the specialist's participation rate should be increasing as the trading volume decreases.

as in Proposition 4 continue to hold even when the specialist is restricted in his dealings.

4.2. Restricted specialist dealings

In the previous subsection, the specialist is allowed to trade freely, with or against the market trend. However, as discussed in introduction, the NYSE restricts the specialist from trading with the market trend (see Rule 104—"Dealings by Specialists"). In this subsection, we investigate implications of such a regulation for the specialist's participation decision. Specifically, we impose a restriction on the specialist's optimization problem by requiring that $yd \leqslant 0$. For example, when the net market order is to buy (d is positive), the specialist is only allowed to sell (i.e., $y \leqslant 0$). We refer to this market organization as the RD (restricted specialist dealings) market.

Proposition 5 (Restricted specialist dealings). *In the RD market there exists unique linear equilibrium. In the equilibrium, the insider's trading strategy and the specialist's pricing rule are, respectively,*

$$x(v) = \beta(v - v_0), \quad p(\alpha, d) = v_0 + \lambda(\alpha)d,$$

where $\lambda(\alpha)$, β and $f(\alpha)$ satisfy the following equations:

$$\lambda(\alpha) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\theta}\right)\chi_\theta + \frac{1}{\alpha}(1 - \chi_\theta), \tag{8}$$

$$\frac{1}{\beta} = 2\mathrm{E}[\lambda(\alpha)], \tag{9}$$

$$\mathrm{E}\left[\frac{1}{4}\left(\frac{1}{\alpha} - \frac{\alpha}{\theta^2}\right)\chi_\theta + \left(\frac{1}{\alpha} - \frac{1}{\theta}\right)(1 - \chi_\theta)\right] = 0, \tag{10}$$

where, as defined in the previous subsection, $\theta = (\beta^2 + \rho^2)/\beta$, and χ_θ is an indicator function taking a value of 1 for $\alpha \leqslant \theta$ and zero otherwise.

From Proposition 5, when the realized depth α is greater than or equal to a critical value θ, the specialist does not participate and the pricing rule is completely determined by the depth of the limit book. On the other hand, when $\alpha < \theta$, the specialist trades for his own account by buying and selling against the market trend. Because the specialist provides extra liquidity to the market, the pricing rule is less steep. Note that, when $\alpha < \theta$, the specialist's strategy is qualitatively the same as that in the no-restriction case, except that the critical value θ for the RD market is different from that for the UD market. It is also easy to check that, in the special case when there is no uncertainty in the limit book ($s = 1$), the equilibrium in Proposition 5 coincides with the CB market equilibrium in Proposition 1.

To investigate the properties of the equilibrium in more detail, let us make the same two-point distribution assumption on α as in the previous subsection. By Proposition 5, there always exists unique equilibrium. The equilibrium is characterized by the critical value θ, which, for all $a>1$, lies between the two realizable values of α, that is, $c/a<\theta<ca$. This means that the specialist participates, with probability of 1/2, when $\alpha = ca$ and does not when $\alpha = c/a$. The critical value is given by $\theta = cr$, where[17]

$$c = \rho \frac{1/a + a/2 + 1/2r}{\sqrt{ar/2 + r/a - 1/2}} \quad \text{and} \quad r = \frac{2a + \sqrt{5a^2+4}}{a^2+4}. \tag{11}$$

The next proposition states comparative statics with respect to exogenous parameters ρ and a.

Proposition 6 (Comparative statics for the RD market). *Along the equilibrium path*

$$\frac{\partial \beta}{\partial \rho} > 0, \quad \frac{\partial \mathrm{E}[\alpha]}{\partial \rho} > 0, \quad \frac{\partial \mathrm{E}[\lambda(\alpha)]}{\partial \rho} < 0.$$

Moreover, under the two-point distribution assumption,

$$\frac{\partial \beta}{\partial a} < 0, \quad \frac{\partial \mathrm{E}[\lambda(\alpha)]}{\partial a} > 0.$$

The comparative statics results in Proposition 6 are qualitatively similar to those in Proposition 3. As ρ increases, the insider becomes more aggressive, the expected depth of the limit book increases, and the expected sensitivity of the pricing rule decreases. As the depth uncertainty a increases, the insider trades less aggressively and the pricing rule is more steep on average.[18]

By Proposition 5 the specialist's participation rate is

$$R(\alpha, d) = \alpha\lambda(\alpha) - 1 = \frac{1}{2}\left(\frac{\alpha}{\theta} - 1\right)\chi_\theta.$$

The above expression reflects the fact that the specialist's participation rate with the market trend is zero. Under the two-point distribution assumption, the expected participation rate against the market trend is

$$\bar{R}_- = \frac{1}{2}\left(1 - \frac{1}{ar}\right) = \frac{1}{2}\left(3 - \sqrt{5 + \frac{4}{a^2}}\right), \tag{12}$$

[17] r is the solution of the following equation:

$$\left(a - \frac{1}{ar^2}\right) + 4\left(\frac{1}{a} - \frac{1}{r}\right) = 0,$$

subject to additional condition that $1/a<r<a$.

[18] In Proposition 3 for the UD market, comparative statics with respect to the depth uncertainty have been proved for *general* distributions of α. For the RD market, similar results for β and $\mathrm{E}[\lambda(\alpha)]$ also hold, but to prove them we rely on the two-point distribution assumption. Moreover, in the RD market, the relationship between the expected depth $\mathrm{E}[\alpha]$ and a may be ambiguous. In particular, under the two-point distribution assumption, when the uncertainty of the limit book is low, $\mathrm{E}[\alpha]$ slightly decreases with a, but when the uncertainty of the limit book is sufficiently high, $\mathrm{E}[\alpha]$ increases with a.

which monotonically increases with a from 0 (when $a = 1$) to approximately 38% (when a approaches infinity). Regardless of the restriction, the specialist's participation rate against the market trend still increases with the uncertainty of the limit book depth. The intuition is similar to that for the UD case. When the book is more uncertain, then (1) the specialist's informational advantage is higher, and (2) the insider is less aggressive. The two effects imply that the specialist trades more aggressively.

Furthermore, the same relationships as in Proposition 4 also hold in the RD market.

Proposition 7. *In the RD market equilibria under the two-point distribution assumption*,

$$\frac{\partial \mathrm{var}(p)}{\partial \bar{R}_-} > 0, \quad \frac{\partial \mathrm{E}[|d|]}{\partial \bar{R}_-} < 0.$$

Proposition 7 will be discussed in more detail in Section 6.

5. Comparison of trading systems

In the previous section, we have examined optimal responses of market participants under two different trading systems, the UD market and RD market. By comparing relative merits of the two systems we can assess the importance of the NYSE regulation that prohibits the specialist from trading with the market trend. Throughout this section, we maintain the assumption that the depth of the limit book α is given by a two-point distribution. The analytical results of this section are illustrated in Fig. 1, where various equilibrium outcomes for the two systems are shown as functions of the depth uncertainty a.

We first compare the sensitivity of the insider's strategy β under the two systems.

Proposition 8. *In linear market equilibria under the two-point distribution assumption, for all* $a > 1$,

$$\beta^{\mathrm{UD}} < \beta^{\mathrm{RD}}.$$

The intuition for Proposition 8 is clear: when the specialist is prohibited from dealing with the market trend, his informational advantage is lower than it is without the regulation, and the insider becomes more aggressive, resulting in a higher value of β (see Fig. 1). This proposition enables us to compare the two trading systems in terms of the posterior variance and the expected sensitivity of pricing rule:

$$\mathrm{var}(v \mid p, d) = \frac{\rho^2}{\rho^2 + \beta^2}\sigma_v^2 \quad \text{and} \quad \mathrm{E}[\lambda(\alpha)] = \frac{1}{2\beta}.$$

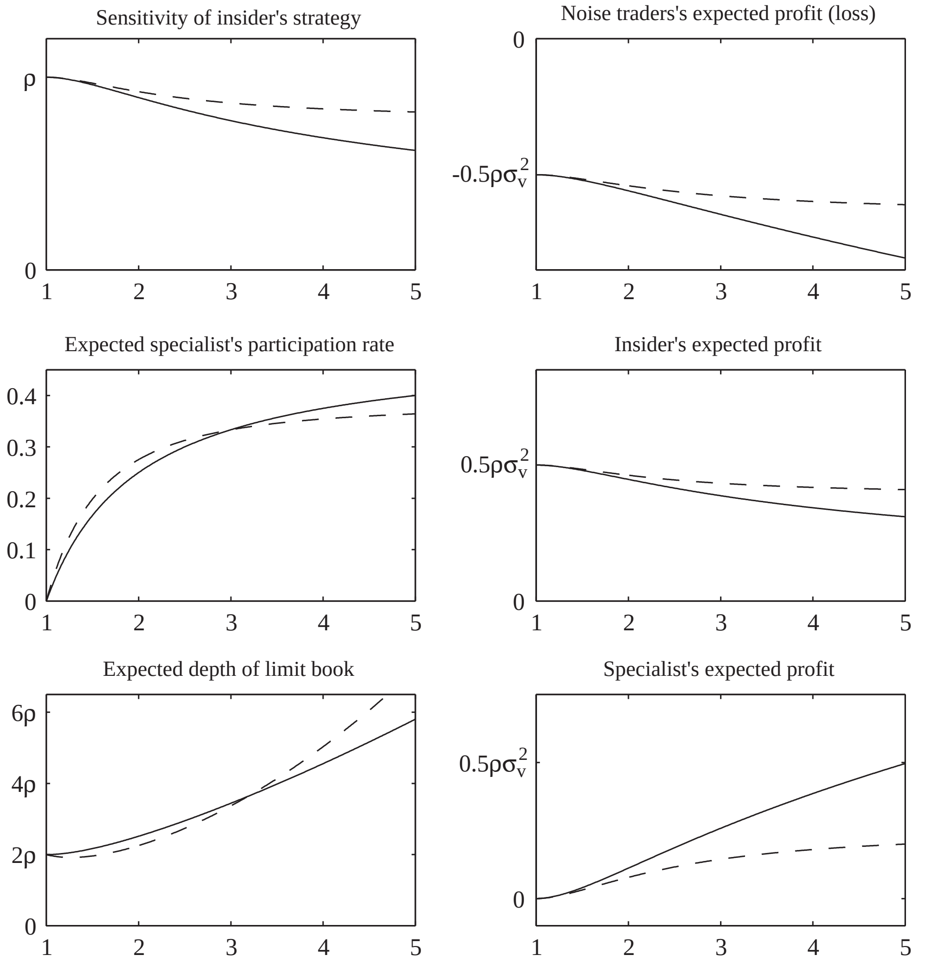

Fig. 1. This figure compares the UD market (the solid line) and the RD market (the dashed line) under the two-point distribution assumption. For different values of the limit book uncertainty a, the figure plots the following equilibrium outcomes. On the left side: the sensitivity of the insider's strategy β, the expected specialist's participation rate (against the market trend) $\bar{R}_{-}$, and the expected depth of the limit book E[α]. On the right side: noise traders' expected profit π_{n}, the insider's expected profit π_{i}, and the specialist's expected profit π_{s}.

The posterior variance can be interpreted as an inverse measure of the price informativeness after the trades are completed and the asset price and trading volume are observed. From Proposition 8, the ex post price is more informative and the average sensitivity of the pricing rule is less when the specialist is not allowed to trade with the market trend. Specifically, for all $a > 1$,

$$\mathrm{var}(v \mid p, d)^{\mathrm{UD}} > \mathrm{var}(v \mid p, d)^{\mathrm{RD}} \quad \text{and} \quad \mathrm{E}[\lambda(\alpha)]^{\mathrm{UD}} > \mathrm{E}[\lambda(\alpha)]^{\mathrm{RD}}.$$

The second inequality is important because it can be used for comparison of bid–ask spreads under the two systems. Even though in our linear model, there are no explicit bid–ask spreads, the sensitivity $\lambda(\alpha)$ can be interpreted as a measure of the *effective* bid–ask spread implicit in the specialist's pricing schedule. (Sensitivity $\lambda(\alpha)$ is the change in the asset's price per a market order of a unit size and is a measure of the cost of trading via market orders.) Under such an interpretation, our model predicts that the bid–ask spread should on average be smaller when the specialist is prohibited from trading with the market trend.

Next, for the two trading systems we look at the specialist's participation rate $\bar{R}_-$ and the expected depth of the limit book $\mathrm{E}[\alpha]$. From the previous sections we already know that, whether specialist is unrestricted or restricted, his participation rate $\bar{R}_-$ is monotonically increasing in a (see Fig. 1). Moreover, using (5) and (12), we verify that

$$\bar{R}_-^{\mathrm{RD}} > \bar{R}_-^{\mathrm{UD}} \quad \text{for } 1 < a < 3,$$
$$\bar{R}_-^{\mathrm{RD}} < \bar{R}_-^{\mathrm{UD}} \quad \text{for } 3 < a.$$

In other words, the specialist's participation rate is higher in the RD market than in the UD market *unless* the uncertainty of the limit book depth is very high.[19] Fig. 1 also reveals that the expected depth of the limit book $\mathrm{E}[\alpha]$ in the RD market is less than in the UD market, again unless the uncertainty of the limit book depth becomes very high (when a is about 3.2 or higher). It may seem counterintuitive that for small a the specialist's participation rate $\bar{R}_-$ is higher and that the expected depth of the limit book $\mathrm{E}[\alpha]$ is lower in the RD market than in the UD market. After all, one might expect that limit trader would *always* submit more orders when the specialist is restricted than when he is not.

It turns out that the regulation has two opposite effects on limit traders. On the one hand, when the realized depth of the limit book is high, the specialist in the RD market is no longer allowed to profit at expense of limit traders. This attracts more limit orders. On the other hand, since the regulation reduces the specialist's informational advantage, the insider now trades more aggressively and the adverse-selection problem due to her private information is more severe. This discourages limit traders from supplying liquidity. When the limit book is not too uncertain, the second effect dominates the first one resulting in lower expected depth and higher specialist's participation in the RD market. In other words, it is the insider (not limit traders) who benefits from the regulation when the uncertainty of the limit book is not too high.[20] However, when the uncertainty is very high, the first effect starts to

[19] For $a \geqslant 3$, the limit book may be interpreted as highly volatile. In particular, when $a = 3$ for the two-point distribution, the coefficient of variation (the standard deviation over the mean) is 80%. For distributions of a positive random variable in general, such a value of the coefficient of variation corresponds to highly "spread out" distributions.

[20] It is interesting to note that the expected depth $\mathrm{E}[\alpha]$ in the RD market is not even monotonically increasing with a, in contrast to that in the UD market. One can see from Fig. 1 that, in the RD market, $\mathrm{E}[\alpha]$ first decreases slightly but then starts to increase (when a is greater than about 1.3). See also footnote 18.

dominated the second one. In this case, the expected depth of the limit book is higher and the specialist's participation is lower in the RD market than in the UD market.

Finally, we compare the two trading systems in terms of welfares of the three types of noncompetitive traders: noise traders, the insider and the specialist. The general formulas for the expected profit of noise traders, the insider and the specialist are

$$\begin{aligned}\pi_{\mathrm{n}} &:= \mathrm{E}[(v - p(\alpha, d))z] = -\frac{\rho^2}{2\beta}\sigma_v^2,\\ \pi_{\mathrm{i}} &:= \mathrm{E}[(v - p(\alpha, d))x] = \frac{\beta}{2}\sigma_v^2,\\ \pi_{\mathrm{s}} &:= \mathrm{E}[(v - p(\alpha, d))(\alpha(p(\alpha, d) - v_0) - d)] = \frac{1}{2\beta}(\rho^2 - \beta^2)\sigma_v^2.\end{aligned}$$

Comparing the UD and RD markets, noise traders and the insider are better off in the RD market while the specialist is better off in the UD market (see Fig. 1). Specifically, for all values of $a > 1$, the following inequalities hold:

$$\pi_{\mathrm{n}}^{\mathrm{UD}} < \pi_{\mathrm{n}}^{\mathrm{RD}}, \qquad \pi_{\mathrm{i}}^{\mathrm{UD}} < \pi_{\mathrm{i}}^{\mathrm{RD}}, \qquad \pi_{\mathrm{s}}^{\mathrm{UD}} > \pi_{\mathrm{s}}^{\mathrm{RD}}.$$

Overall, the RD appears to have several advantages over the UD market. In particular, under the NYSE regulation that prohibits the specialist from trading with the market trend, noise traders' welfare is higher, the price is more informative, and the specialist's pricing schedule is less steep (implying smaller bid–ask spreads). Clearly, the liquidity demanders (the insider and noise traders) benefit when the regulation is imposed. On the other hand, the specialist is better off in the market without the regulation. Note that, because our model is inherently a zero sum game, moving from one market organization to the other cannot benefit every player.

Our model has also an important policy implication regarding how equilibrium outcomes depend on the uncertainty of the limit book depth. From Propositions 3 and 6, as the limit book becomes less uncertain (a decreases), the sensitivity of the insider's strategy increases. This in turn implies that the market order traders (the insider and noise traders) benefit from the limit book being less uncertain. In particular, regardless of the regulation, the ex post price is more informative, the price rule is on average less steep (bid–ask spreads are smaller), and the expected profits of noise traders and the insider are larger as a decreases. On the other hand, the expected profit of the specialist becomes smaller as a decreases. See the comparison of the two market organizations in Fig. 1. In the limiting case of the certain book $a = 1$, the insider is the most aggressive ($\beta = \rho$), the bid–ask spreads are the lowest, and the welfares of the market order traders are the highest. As for the specialist, he does not participate in trading and his profit is zero.

These results are very intuitive. When the uncertainty of the limit book is high, the information asymmetry about the depth grows in favor of the specialist over the other traders and the specialists extracts greater profits from both noise traders and the insider. In contrast, when a is low, the specialist's knowledge of the realized depth is less valuable. As a result, the insider is more aggressive and is able to more fully capitalize on her private information. Noise traders' welfare improves because their expected loss to the specialist becomes smaller.

The conclusion that the market order traders prefer the certain limit book is also consistent with Glosten (1994) and Baruch (1997). In his Proposition 6, Glosten compares two market systems. In the first system, there is no specialist and all liquidity is supplied by a completely open (transparent) limit book. In the second system, there is no limit book and all liquidity is supplied by a monopolistic specialist as in Glosten (1989). He finds that risk averse market order traders are always better off with the limit book. Baruch compares specialist's systems with open and closed limit books. In his model, liquidity is supplied by a number of limit order traders who imperfectly compete in linear price schedules. When the limit book is open, it has a constant depth but, when the limit book is closed, its depth is uncertain because the number of limit order traders is random. Baruch formulates a sufficient condition under which the market order traders prefer the open limit book.[21]

Similarly, in our model, the market order traders always prefer as certain limit book as possible. We find that this conclusion holds even under the NYSE regulation, which prohibits the specialist to extract liquidity from the market. The trading costs of traders who demand immediacy are the lower, the less uncertain the limit book. It is important to emphasize, however, that in our model a *certain* book is not exactly the same as an *open* book, although the two are closely related. A book can be completely open but if it changes frequently then limit order traders will view the book as at least somewhat uncertain (corresponding to depth uncertainty parameter s close but not exactly equal to 1). A certain book is a book where, all profit opportunities from liquidity provision are fully exploited at any time. This, however, might not be easy to achieve in practice, especially for continuous markets. For example, an open book will have a somewhat random depth if there are noise limit traders and if limit traders do not monitor the limit book constantly or there is a delay before a limit order can reach the exchange. Moreover, as noted in an earlier section, even in officially transparent limit order systems, such as the Paris Bourse and the Toronto Stock Exchange Computer Assisted Trading System (CATS), there are inevitably hidden orders in the limit books so that the true depth of the book is not completely known to all traders (see Biais et al., 1995).[22]

[21] In Baruch (1997), there is a strategic aspect in limit order traders' behavior because each trader knows his own price schedule and thus possesses some information on the book's contents. This implies that (1) limit order traders earn strictly positive expected profits, and (2) the equilibrium may fail to exist if there is not enough competition between limit traders.

[22] The above discussion also implies that, certain and uncertain book markets are not two *different* market organizations with identical parameters. Instead, a certain book is merely the limiting case of an uncertain book when exogenous parameter s approaches 1. In other words, when comparing certain and uncertain book markets, we compare different economies, as opposed to different market organizations.

6. Empirical issues

Our analytical framework has a number implications for empirical work on the determinants of specialist's trading, across stocks and over time.

First, regardless of the regulation, the specialist's participation rate is positively related to price volatility and negatively related to average trading volume. See Propositions 3 and 6. These theoretical predictions are supported by recent empirical studies. Madhavan and Sofianos (1998) document that across the NYSE specialists, the participation rate is (significantly) positively related to stock price volatility and negatively related to trading volume, even after controlling for stock market capitalization. They attribute the finding that specialists are more active in volatile stocks to specialists' affirmative obligation to stabilize prices.

In particular, Madhavan and Sofianos (1998) state that "Price continuity rules require specialists to trade to stabilize prices, suggesting that the participation will be higher in stocks where intraday return volatility is large". In our model, we do not impose the price continuity rule (although it is related to the restriction which prohibits trading with the market trend). An interesting implication of Proposition 4 is that the positive relationship in (6) holds *even when* the specialist is not constrained by the price continuity rule. In other words, when the price volatility is high, the specialist *voluntarily* increases his participation, as a result of his optimal behavior and not as a result of the restrictions imposed by the exchange.

The negative relationship between the specialist's participation and volume is also consistent with empirical findings by Sofianos (1995) and Chung et al. (1999). Sofianos do not explicitly discuss this negative relationship; however, it can clearly be seen in his Table 6. Chung et al. document that the NYSE specialists tend to quote more actively for low-volume stocks.

Second, one of the main predictions of our model is that the specialist's participation rate and expected profit both increase with the uncertainty of the depth of the limit book. See (5) and (12), and the discussion in Section 5. To test this prediction, one would need to look at specialists' participation rates for stocks that have "certain" and "uncertain" limit order books. Furthermore, there are times when the limit order books are quite variable and times when they are quite stable, implying that the specialist participates more in the former case than in the latter. Interestingly, Chung et al. document that the NYSE specialists tend to quote more actively during early hours of trading, when there are fewer limit orders submitted and when the resulting limit book is presumably more uncertain.

Another cross-sectional prediction of our model is that the average specialist's participation rate does *not* depend on ρ, the degree of adverse-selection problem. This means that, other things being equal, the specialist's participation should not systematically differ whether the amount of private information in the market is high or low.

Third, the average sensitivity of pricing rule $E[\lambda(\alpha)]$ increases with the uncertainty of the limit book (see Propositions 3 and 6). As discussed in Section 5, this implies that the average bid–ask spread is also positively related to the uncertainty of the limit book.

Fourth, for transaction-level studies, our model predicts that the specialist provides more liquidity when the realized depth of the limit book is low. An empirical support can be found in Kavajecz and Odders-White (2001). They use the depth of limit book as an explanatory variable for the specialist's participation behavior and find that the depths of bid/ask quotes significantly affect the specialist's pricing schedules.

Fifth, the model can be useful for analysis of the specialist's trading using recently available order flow data. Suppose that an empiricist studies the RD market. In this market, the specialist's participation is related to the realized depth of the limit book, with the participation rate decreasing from 50% to 0% as the realized depth α increases from 0 to the critical value θ; when the depth is greater than θ, the specialist does not trade at all. This means that the empiricist can fit the model with only one unobservable parameter θ (the realized depth of the limit book α can be reasonably well estimated using the data on composition of the limit order book). Furthermore, the critical value $\theta = (\beta^2 + \rho^2)/\beta$ increases with the noise-to-asset-volatility ratio ρ and decreases with the uncertainty of the limit book, the relationships that are testable across stocks.

Finally, recent developments in the NYSE may provide additional testable predictions of the model.[23] In March 2001, the NYSE launched a new information system called NYSE MarkeTrac, which offers "a virtual 3-D connection to the point-of-sale". More importantly, in the near future NYSE MarkeTrac will be enhanced with another feature, called NYSE OpenBook, which will provide a real-time view at the specialist limit order book. Our model suggests that the NYSE's move to open the limit book will cause the following across-the-board impacts: (1) a decline in the specialist's participation and profits, (2) a decline in the bid–ask spreads, (3) an increase in the trading volume, and (4) a decline in the price volatility. These predictions will certainly be testable.

7. Conclusion

We have examined specialist's participation in a security market in the presence of competition from limit order traders. The trading systems in this paper extend the classic model of Kyle (1985) by allowing the specialist to execute incoming market orders against limit orders as well as his own inventory. The limit book in our model has a random depth, observable to the specialist but unobservable to other traders. We find that when the limit book is certain, the specialist does not participate; when the book is uncertain, the specialist has incentives to trade for his own account with or against the market trend, depending upon the realized depth of the limit book.

[23] We are grateful to the referee for this suggestion.

In particular, we show that the specialist wishes to trade against the market trend when the realized depth of the limit book is too low and to trade with the market trend when it is too high. That is, the specialist has incentives to provide extra liquidity to the market when the depth of the limit book is less than some critical value, but he also has incentives to consume liquidity from the limit book when the depth of the limit book is greater than the critical value.

To be consistent with current practices of the NYSE, we investigate implications of the regulation that restricts the specialist from trading with the market trend. This regulation successfully captures the positive aspect of the specialist system: the specialist is allowed to supply liquidity to the market, but he is prohibited to extract the liquidity from the market. We find that the market order traders (noise traders and the insider) benefit from the regulation. In particular, under the regulation, the costs of trading via market orders decrease, noise traders' welfare improves, the insider is more aggressive in submitting her market orders, and the security prices are more informative. We also argue that the market order traders are better off when the limit book is less uncertain. This could be achieved by making the limit book as transparent to the public as possible.

Whether the regulation is in effect or not, the specialist's participation rate is positively related to the asset price volatility and is negatively related to the trading volume. These relationships arise because the specialist's participation rate and the price volatility increase while the trading volume decreases as the limit book depth becomes more uncertain. These results provide formal theoretical justifications to empirical findings in Madhavan and Sofianos (1998), Sofianos (1995), and Chung et al. (1999).

Appendix A

A.1. Assumptions

We assume that distribution of limit book depth α is drawn from family of distributions $\mathscr{D}$ which satisfies the following two assumptions.

Assumption 1. For any distribution $f(\alpha)\in\mathscr{D}$, the following finite moments exist: $\mathrm{E}[\alpha]$, $\mathrm{E}[1/\alpha]$, and $\mathrm{E}[1/\alpha^2]$.

Assumption 2. Distributions in $\mathscr{D}$ are uniquely parameterized by their mean μ and statistic s. That is, for all $\mu\in(0,\infty)$ and $s\in(1,\infty)$ there exists unique distribution $f(\alpha)\in\mathscr{D}$ such that $\mathrm{E}[\alpha]=\mu$ and $(\mathrm{E}[\alpha]\mathrm{E}[1/\alpha])^{1/4}=s$.

The first assumption allows us to compute various equilibrium outcomes. The second assumption ensures that a linear equilibrium in Theorems 2 and 3 always exists and is unique. This assumption is satisfied by many popular families of two-parameter distributions of positive random variables, including two-point distributions, lognormal distributions, Gamma distributions, and others. We illustrate Assumption 2 with several examples below.

1. Let $\mathscr{D}$ be the family of two-point distributions with parameters $c>0$ and $a\geqslant 1$, such that α can take on two values ca and c/a with equal probabilities. This family of distributions is used in the main text to explicitly compute various equilibrium outcomes. It is uniquely parameterized by $\mu \in (0,\infty)$ and $s \in (1,\infty)$ where

$$\mu = \mathrm{E}[\alpha] = \frac{c}{2}\left(a+\frac{1}{a}\right), \quad s = \left(\mathrm{E}[\alpha]\mathrm{E}\left[\frac{1}{\alpha}\right]\right)^{1/4} = \sqrt{\frac{1}{2}\left(a+\frac{1}{a}\right)}.$$

2. Let $\mathscr{D}$ be the family of lognormal distributions with parameters $\eta \in (-\infty,\infty)$ and $\sigma \in (0,\infty)$ such that

$$f(\alpha) = \frac{1}{\sqrt{2\pi}\sigma}\frac{1}{\alpha}\,\mathrm{e}^{-(\ln\alpha-\eta)^2/2\sigma^2}, \quad \alpha>0.$$

This family of distributions can be uniquely parameterized by $\mu \in (0,\infty)$ and $s \in (1,\infty)$ where

$$\mu = \mathrm{E}[\alpha] = \mathrm{e}^{\eta+\sigma^2/2}, \quad s = \left(\mathrm{E}[\alpha]\mathrm{E}\left[\frac{1}{\alpha}\right]\right)^{1/4} = \mathrm{e}^{\sigma^2/4}.$$

3. Let $\mathscr{D}$ now be the family of Gamma distributions with parameters $a>1$ and $b>0$, such that

$$f(\alpha) = \frac{b^a}{\Gamma(a)}\alpha^{a-1}\,\mathrm{e}^{-b\alpha}, \quad \Gamma(a) = \int_0^\infty \alpha^{a-1}\,\mathrm{e}^{-\alpha}\,\mathrm{d}\alpha, \;\; \alpha>0.$$

Then this family of distributions can be uniquely parameterized by $\mu \in (0,\infty)$ and $s \in (1,\infty)$ where

$$\mu = \mathrm{E}[\alpha] = \frac{a}{b}, \quad s = \left(\mathrm{E}[\alpha]\mathrm{E}\left[\frac{1}{\alpha}\right]\right)^{1/4} = \sqrt{\frac{a}{a-1}}.$$

A.2. Proofs

Proof of Proposition 1. Let $p(d)=\delta+\lambda d$. The insider's profit maximization problem is

$$\max_x\ \mathrm{E}[(v-p(x+z))x \mid v] = \max_x\ \mathrm{E}[(v-\delta-\lambda x)x \mid v].$$

From the first-order condition (FOC), the insider's strategy is a linear function $x(v)=\gamma+\beta v$ with constants β and γ given by

$$\beta = 1/(2\lambda), \quad \gamma = -\delta\beta.$$

The insider's second-order condition (SOC) is satisfied if $\lambda > 0$. Consider now the specialist's profit maximization problem:

$$\max_p \mathrm{E}[(v-p)(\alpha(p-v_0)-d)\,|\,d] = \max_p(\mathrm{E}[v\,|\,d]-p)(\alpha(p-v_0)-d).$$

By the FOC for the above problem,

$$p(d) = \frac{1}{2}\left(\mathrm{E}[v\,|\,d] + v_0 + \frac{d}{\alpha}\right),$$

and the SOC is simply $\alpha > 0$. Because of the normality of v and z

$$\mathrm{E}[v\,|\,d] = \mathrm{E}[v\,|\,d = \gamma + \beta v + z] = \frac{v_0\rho^2 + (d-\gamma)\beta}{\rho^2+\beta^2}.$$

Using $\gamma = -\delta\beta$ we can write

$$\delta = v_0, \quad 2\lambda = \frac{1}{\alpha} + \frac{\beta}{\rho^2+\beta^2}.$$

Since $\beta = 1/(2\lambda)$, we obtain

$$\frac{1}{\beta} = \frac{1}{\alpha} + \frac{\beta}{\rho^2+\beta^2}$$

or

$$(\alpha-\beta)\rho^2 = \beta^3.$$

Note that, for any positive α, the last equation has exactly one positive solution of β. Finally, the zero profit condition for limit orders is

$$\mathrm{E}[(v-p(d))\alpha(v_0-p(d))] = \alpha\mathrm{E}[\lambda d(v-v_0-\lambda d)] = \alpha\sigma_v^2\left(\lambda^2\rho^2 - \frac{1}{4}\right).$$

Setting the expected profit to zero yields $\lambda = 1/(2\rho) > 0$, and thus $\beta = \rho$, and $\alpha = 2\rho$. $\square$

Proof of Proposition 2. The proof consists of two parts. The first part derives the necessary and sufficient conditions of a linear market equilibrium. The second part shows that under Assumption 2, there always exists unique linear equilibrium.

Part 1. This part is a modification of the proof for Proposition 1. Let the specialist's pricing rule be $p(\alpha, d) = \delta + \lambda(\alpha)d$. Then the insider's optimal strategy must be $x(v) = \gamma + \beta v$ with constants β and γ such that

$$\frac{1}{\beta} = 2\mathrm{E}[\lambda(\alpha)], \quad \gamma = -\delta\beta.$$

The insider's SOC is $\mathrm{E}[\lambda(\alpha)0$. From the specialist's FOC we obtain that

$$\delta = v_0, \quad 2\lambda(\alpha) = \frac{1}{\alpha} + \frac{\beta}{\rho^2+\beta^2},$$

where $\alpha > 0$ by the specialist's SOC. Let $\theta = (\rho^2 + \beta^2)/\beta$, then we can write the limit order traders' expected profit as

$$E[\alpha(v - p(\alpha, d))(v_0 - p(\alpha, d))] = \frac{\beta\sigma_v^2}{4}\left(\theta \mathrm{E}\left[\frac{1}{\alpha}\right] - \frac{1}{\theta}\mathrm{E}[\alpha]\right).$$

In an equilibrium, the limit book expected profit is zero:

$$\theta \mathrm{E}\left[\frac{1}{\alpha}\right] = \frac{1}{\theta}\mathrm{E}[\alpha] = s^2.$$

On the other hand, from the insider's FOC,

$$\theta \mathrm{E}\left[\frac{1}{\alpha}\right] = \frac{\rho^2}{\beta^2},$$

which yields the following necessary and sufficient conditions of a linear equilibrium:

$$\beta = \frac{\rho}{s}, \quad \mathrm{E}[\alpha] = \rho(s + s^3), \quad \mathrm{E}[\lambda(\alpha)] = \frac{1}{\alpha} + \frac{1}{\rho}\frac{s}{s^2 + 1}. \tag{13}$$

Part 2. Note that, under Assumption 2, when s is exogenously given, the equilibrium distribution of the limit book depth $f(\alpha)$ is solely determined its mean $\mu(= \mathrm{E}[\alpha])$. For any given pair of exogenous parameters (ρ, s), equations in (13) define unique β, $\mathrm{E}[\alpha]$, and $\lambda(\alpha)$, which in turn implies the existence and uniqueness of the linear market equilibrium $(x(v), p(\alpha, d), f(\alpha))$. □

Proof of Proposition 3. The comparative statics obtain immediately from the following explicit representations:

$$\beta = \frac{\rho}{s}, \quad \mathrm{E}[\alpha] = \rho(s + s^3), \quad \mathrm{E}[\lambda(\alpha)] = \frac{s}{2\rho}. \qquad \square$$

Proof of Proposition 4. The price volatility can be written as

$$\mathrm{var}(p) = \mathrm{var}(v_0 + \lambda(\alpha)d) = \mathrm{E}[\lambda(\alpha)^2](\rho^2 + \beta^2)\sigma_v^2.$$

For the UD market, this expression becomes

$$\mathrm{var}(p) = \frac{\sigma_v^2}{4}\left(\frac{1}{s^2}\mathrm{var}\left(\frac{\rho}{\alpha}\right) + 1\right)(1 + s^2).$$

Under the two-point distribution assumption,

$$\mathrm{var}\left(\frac{\rho}{\alpha}\right) = \frac{\rho^2}{c^2}(s^4 - 1) = s^2\frac{s^2 - 1}{s^2 + 1}.$$

After simplification,

$$\mathrm{var}(p) = \tfrac{1}{2}s^2\sigma_v^2.$$

Therefore, for all $a \geqslant 1$

$$\frac{\partial \text{var}(p)}{\partial a} > 0,$$

implying that the price volatility decreases when the limit book becomes more certain. It follows from (5) that

$$\frac{\text{d}\bar{R}_-}{\text{d}a} > 0, \quad \frac{\text{d}\bar{R}_+}{\text{d}a} > 0. \tag{14}$$

Since both $\bar{R}_-$ and $\bar{R}_+$ are invertible functions in a, the following differentiations are well-defined:

$$\frac{\partial \text{var}(p)}{\partial \bar{R}_-} = \frac{\partial \text{var}(p)/\partial a}{\text{d}\bar{R}_-/\text{d}a} > 0, \quad \frac{\partial \text{var}(p)}{\partial \bar{R}_+} = \frac{\partial \text{var}(p)/\partial a}{\text{d}\bar{R}_+/\text{d}a} > 0.$$

To prove the relationships in (7), note that $d = \beta(v - v_0) + z$ is a normal random variable with zero mean and variance of $\beta^2\sigma_v^2 + \sigma_z^2$. Therefore,

$$\text{E}[|d|] = \sqrt{\frac{2\sigma_v^2}{\pi}}\sqrt{\beta^2 + \rho^2}.$$

Since $\beta = \rho/s$, where s is given by Eq. (4), we have $\partial \text{E}[|d|]/\partial a < 0$. The rest then follows from (14). □

Proof of Proposition 5. As with Proposition 2, the proof consists of two parts. The first part derives the equilibrium conditions while the second part proves the existence and uniqueness of the linear equilibrium.

Part 1. Let $p(\alpha, d) = \delta + \lambda(\alpha)d$. As in the proof of Proposition 2, the insider's optimal strategy is $x(v) = \beta(v - \delta)$, with $1/\beta = 2\text{E}[\lambda(\alpha)]$ (the insider's SOC is again $\text{E}[\lambda(\alpha)0)$.

Under the regulation, the specialist solves the constrained optimization problem:

$$\max_p \quad E[(v - p)(\alpha(p - v_0) - d) \mid \alpha, d]$$
$$\text{s.t.} \quad (\alpha(p - v_0) - d)d \leqslant 0.$$

Since the objective function is concave and the constraint is linear in p, the first-order necessary and sufficient conditions are

$$-2\alpha p + \alpha v_0 + d + \alpha E[v \mid d] - \alpha q(\alpha, d)d = 0,$$
$$q(\alpha, d)(\alpha(p - v_0) - d)d = 0,$$
$$(\alpha(p - v_0) - d)d \leqslant 0,$$

where $q(\alpha, d)$ denotes Lagrange multiplier for a given pair (α, d). Recalling that

$$\text{E}[v \mid d] = \frac{v_0\rho^2 + (d + \beta\delta)\beta}{\rho^2 + \beta^2},$$

we obtain

$$p(\alpha, d) = \delta + \lambda(\alpha)d = v_0\left(1 - \frac{\beta^2}{2(\rho^2 + \beta^2)}\right) + \frac{\delta\beta^2}{2(\rho^2 + \beta^2)} + \frac{d}{2}\left(\frac{1}{\alpha} + \frac{\beta}{\rho^2 + \beta^2} - q(\alpha, d)\right).$$

From the last equation, $\delta = v_0$ and

$$\lambda(\alpha) = \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\theta} - q(\alpha, d)\right),$$

where Lagrange multiplier $q(\alpha, d)$ satisfies the following conditions:

$$\frac{\alpha d^2}{2} q(\alpha, d)\left(-\frac{1}{\alpha} + \frac{1}{\theta} - q(\alpha, d)\right) = 0,$$
$$\frac{\alpha d^2}{2}\left(-\frac{1}{\alpha} + \frac{1}{\theta} - q(\alpha, d)\right) \leqslant 0.$$

For $d \neq 0$, function $q(\alpha, d)$ can be written as

$$q(\alpha, d) = \begin{cases} 0 & \text{if } \alpha \leqslant \theta, \\ \frac{1}{2}\left(\frac{1}{\theta} - \frac{1}{\alpha}\right) & \text{if } \alpha > \theta. \end{cases}$$

This implies that the sensitivity of the pricing rule is

$$\lambda(\alpha) = \begin{cases} \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\theta}\right) & \text{if } \alpha \leqslant \theta, \\ \frac{1}{\alpha} & \text{if } \alpha > \theta. \end{cases}$$

Finally, given the realization of depth α, the limit book (conditional) expected profit

$$\Phi(\alpha) = \mathrm{E}[(v - p(\alpha, d))\alpha(v_0 - p(\alpha, d)) \mid \alpha] = \sigma_v^2 \beta\alpha\lambda(\alpha)(\theta\lambda(\alpha) - 1)$$
$$= \sigma_v^2\beta \begin{cases} \frac{1}{4}\left(\frac{\theta}{\alpha} - \frac{\alpha}{\theta}\right) & \text{if } \alpha \leqslant \theta, \\ \left(\frac{\theta}{\alpha} - 1\right) & \text{if } \alpha > \theta. \end{cases}$$

Setting the (unconditional) expected profit of the limit book to zero, $\mathrm{E}[\Phi(\alpha)] = 0$, concludes Part 1 of the proof.

Part 2. Let $\mathscr{M}(\rho, s)$ define a market model with two exogenous parameters, noise-to-asset volatility ratio ρ and limit book uncertainty s. Our goal is to prove that for any $\rho > 0$ and $s \geqslant 1$ there always exists a unique triplet, a sensitivity of the insider's strategy β, a function of the realized limit book depth $\lambda(\alpha)$, and a distribution $f(\alpha)$, which satisfies the system of equation (8)–(10) for market model $\mathscr{M}(\rho, s)$. In view of representation for $\lambda(\alpha)$ in (8) and because distributions of α satisfy Assumption 2, any linear equilibrium is completely characterized by a triplet of constants (β, θ, μ) (which depend on exogenous parameters ρ and s).

We will rely on the fact that system (8)–(10) is homogeneous with respect to ρ. Specifically,

Fact 1. *If triplet* (β, θ, μ) *solves* (8)–(10) *for* $\mathcal{M}(\rho, s)$, *then for all* $k \in (0, \infty)$ *triplet* $(k\beta, k\theta, k\mu)$ *solves* (8)–(10) *for* $\mathcal{M}(k\rho, s)$.

Step 1 (Existence): In view of Fact 1, market model $\mathcal{M}(\rho, s)$ always has a solution if market model $\mathcal{M}(\rho_0, s)$ (with *any* $\rho_0 > 0$ and *the same* s) has a solution. Therefore, this step will be proved, if we can show that there exist positive β_0, θ_0, μ_0, and ρ_0 such that triplet $(\beta_0, \theta_0, \mu_0)$ solves system (8)–(10) for market model $\mathcal{M}(\rho_0, s)$.

To do so, we first arbitrary choose $\mu_0 > 0$. Then, for fixed s and μ_0 (recall that they together determine distribution of book depth, $f(\alpha; \mu_0, s)$) we *uniquely* find the critical value $\theta_0 > 0$ which satisfies Eq. (10). Indeed, let us rewrite Eq. (10) as $\mathrm{E}[I(\theta; \alpha)] = 0$, where

$$I(\theta; \alpha) := \begin{cases} \frac{1}{4}\left(\frac{1}{\alpha} - \frac{\alpha}{\theta^2}\right) & \text{if } \alpha \leqslant \theta, \\ \left(\frac{1}{\alpha} - \frac{1}{\theta}\right) & \text{if } \alpha > \theta. \end{cases}$$

For any two numbers, θ_1 and θ_2 such that $0 < \theta_1 < \theta_2$, we have

$$I(\theta_2; \alpha) - I(\theta_1; \alpha) = \begin{cases} \frac{\alpha}{4}\left(\frac{1}{\theta_1^2} - \frac{1}{\theta_2^2}\right) & \text{if } \alpha \leqslant \theta_1, \\ \frac{1}{4}\left(\frac{1}{\alpha} - \frac{\alpha}{\theta_2^2}\right) - \left(\frac{1}{\alpha} - \frac{1}{\theta_1}\right) & \text{if } \theta_2 < \alpha \leqslant \theta_2, \\ \left(\frac{1}{\theta_1} - \frac{1}{\theta_2}\right) & \text{if } \alpha > \theta_2. \end{cases}$$

In the last equation, all terms in the RHS are positive. Thus, we have $\mathrm{E}[I(\theta_2, \alpha)] - \mathrm{E}[I(\theta_1, \alpha)0$, meaning that $\mathrm{E}[I(\theta, \alpha)]$ is a monotone increasing and continuous function of θ. It is also easy to check that

$$\mathrm{E}[I(0, \alpha)] = -\infty, \quad \mathrm{E}[I(\infty, \alpha)] = \frac{1}{4}\mathrm{E}\left[\frac{1}{\alpha}\right] > 0.$$

Therefore, there exists unique value θ_0 such that $\mathrm{E}[I(\theta_0, \alpha)] = 0$. Finally, from (9) and (10) and definition of the critical value θ we can (again *uniquely*) determine $\lambda_0(\alpha)$, β_0, and ρ_0 as follows:

$$\lambda_0(\alpha) = \begin{cases} \frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\theta_0}\right) & \text{if } \alpha \leqslant \theta_0, \\ \frac{1}{\alpha} & \text{if } \alpha > \theta_0 \end{cases}$$

and

$$\frac{1}{\beta_0} = 2\mathrm{E}[\lambda_0(\alpha)], \quad \rho_0^2 = \beta_0(\theta_0 - \beta_0).$$

One problem remains, however. The last equation defines a valid value of ρ_0 only if $\beta_0 < \theta_0$. We now show that this inequality is indeed always satisfied. We use the following representation:

$$2\lambda_0(\alpha) - 4I(\theta_0;\alpha) - \frac{1}{\theta_0} = \begin{cases} \dfrac{\alpha}{\theta_0^2} & \text{if } \alpha \leqslant \theta_0, \\ \left(\dfrac{3}{\theta_0} - \dfrac{2}{\alpha}\right) & \text{if } \alpha > \theta_0. \end{cases}$$

Since the two terms of the RHS are positive, after taking expectations, we obtain inequality

$$0 < \frac{1}{\beta_0} - 4\mathrm{E}[I(\theta_0,\alpha)] - \frac{1}{\theta_0} = \frac{1}{\beta_0} - \frac{1}{\theta_0}.$$

Therefore, $\beta_0 < \theta_0$ and required β_0, θ_0, and ρ_0 always exist (and are *uniquely* determined) for all s and μ_0.

Step 2 (Uniqueness): Assume that market model $\mathcal{M}(\rho,s)$ has two solutions, (β_1,θ_1,μ_1) and (β_2,θ_2,μ_2), and let $k=\mu_1/\mu_2$. By Fact 1, triplet $(k\beta_2,k\theta_2,k\mu_2)$ is a solution for $\mathcal{M}(k\rho,s)$. Since $k\mu_2=\mu_1$, from the proof of Step 1 it follows that $k=1$ and that the two solutions must coincide. □

Proof of Proposition 6. From the proof of Proposition 5, for all pairs of exogenous parameters $\rho > 0$ and $s \geqslant 1$ there always exists a unique triplet, which is completely characterized by β, θ, and μ, which are (single-valued) functions of ρ and s. Furthermore, in view of Fact 1, these functions are homogeneous of degree one with respect to ρ. That is, for all $k \in (0,\infty)$

$$\beta(k\rho,s) = k\beta(\rho,s), \quad \theta(k\rho,s) = k\theta(\rho,s), \quad \mu(k\rho,s) = k\mu(\rho,s).$$

Comparative statics with respect to ρ then follow immediately.

To prove comparative statics with respect to a, note that, under the two-point distribution,

$$\beta^2 = \frac{2a\rho^2}{a^2r + 2r - a},$$

where r is as in Eq. (11). Elementary but tedious computations show that $\partial\beta/\partial a < 0$. In an equilibrium, the expected sensitivity of the pricing rule is $\mathrm{E}[\lambda(\alpha)] = 1/(2\beta)$, implying that $\partial\mathrm{E}[\lambda(\alpha)]/\partial a > 0$. □

Proof of Proposition 7. The proof is similar to that of Proposition 4. First, from Eq. (12), for the RD market

$$\frac{\mathrm{d}\bar{R}_-}{\mathrm{d}a} > 0 \quad \text{for all } a \geqslant 1.$$

Under the two-point distribution assumption, the price volatility in the RD market is

$$\mathrm{var}(p) = \mathrm{E}[\lambda(\alpha)^2](\rho^2 + \beta^2)\sigma_v^2 = \frac{r}{2}\sigma_v^2 \frac{(a/2 + 1/2r)^2 + 1/a^2}{a/2 + 1/2r + 1/a},$$

where r is given in Eq. (11). Straightforward but tedious algebra shows that, for all $a \geqslant 1$,

$$\frac{\partial \mathrm{var}(p)}{\partial a} > 0$$

and therefore

$$\frac{\partial \mathrm{var}(p)}{\partial \bar{R}_-} = \frac{\partial \mathrm{var}(p)/\partial a}{\mathrm{d}\bar{R}_-/\mathrm{d}a} > 0.$$

Similar to the proof of Proposition 3, the second part of this proposition follows from the facts that $\mathrm{E}[|d|]$ increases with β and that $\partial\beta/\partial a < 0$ (by Proposition 6). Since $\mathrm{d}\bar{R}_-/\mathrm{d}a > 0$, we have $\partial \mathrm{E}[|d|]/\partial \bar{R}_- < 0$. □

Proof of Proposition 8. We want to show that, under the two-point distribution assumption, for all $a > 1$

$$\beta^{\mathrm{UD}} = \frac{\rho}{\sqrt{\frac{1}{2}(a + 1/a)}} < \beta^{\mathrm{RD}} = \frac{\rho}{\sqrt{ar/2 + r/a - 1/2}},$$

where r is given in Eq. (11). The claim will follow if we prove that

$$\frac{a^2 + 2}{a^2 + a + 1} < \frac{1}{r} = \sqrt{5a^2 + 4} - 2a$$

or, equivalently, that

$$(a^2 + 2 + 2a(a^2 + a + 1))^2 < (5a^2 + 4)(a^2 + a + 1)^2.$$

After regrouping and collecting terms, the last inequality becomes

$$0 < a^2(a - 1)^2(a^2 + 1)$$

and we are done. □

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