Optimal Life Insurance Purchase, Consumption and Investment ⋆

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Abstract

Merton's famous continuous-time model of optimal consumption and investment is extended to allow for a wage earner to have a random lifetime and to use a portion of the income to purchase life insurance in order to provide for his or her estate. With constant relative risk aversion utility functions, the wage earner's problem is to find the optimal consumption, investment, and insurance purchase decisions in order to maximize expected utility of consumption, of the size of the estate in the event of premature death, and of the size of the estate at the time of retirement. Dynamic programming methods are used to obtain explicit solutions. Numerical results are presented, from which are drawn economic implications and understanding.

Key words: Life Insurance, Investment/Consumption Model, HJB Equation, CRRA Utilities

1 Introduction

In this paper we consider the optimal life insurance purchase, consumption and portfolio management strategies for a wage earner subject to mortality risk in a continuous time economy. Decisions are made continuously about these three strategies for all time $t \in [0, min(\tau, T)]$, where the fixed planning

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horizon T can be interpreted as the planned retirement time of the wage earner and where τ is the random time when he or she dies.

The wage earner receives his/her income at rate $i(t)$ continuously, but this is terminated by the wage earner's death or retirement, whichever happens first. The wage earner needs to buy life insurance to protect his/her family due to his/her premature death. Needless to say, the bigger the insurance premium paid by the wage earner, the bigger the claim paid to his/her family upon premature death. As will be seen, the insurance policy is like term insurance, but with an arbitrarily small term because the wage earner buys insurance at a continuous rate.

Aside from consumption and life insurance purchase, the wage earner also has the opportunity to invest in the capital market which consists of a riskless security and a risky security. Although we only consider one risky security in this paper, it is easy and not technically difficult to include multiple risky securities in our model.

The problem is to find the strategies that are best in terms of both the family's consumption for all $t \leq min(\tau, T)$ as well as the size of the estate at time $min(\tau, T)$.

Roughly speaking, most of the research leading up to our model tended to ignore one or more important aspects of the problem we consider. As we review the existing literature we will try to point out shortcomings of existing models and the contributions of ours.

In one line of research the emphasis was on the demand for life insurance for an individual with an uncertain lifetime, at the expense of optimal investment decisions. Yarri (1965) might have been the first to consider optimal financial planning decisions for an individual with an uncertain lifetime. His objective, subject to a nonnegative wealth constraint, was to maximize

$$E\left[\int_0^\tau U(c(t))dt\right],$$

where τ, the individual's lifetime, is a bounded random variable that was assumed to take values in $[0, T]$ for some fixed, positive number T, where $c(t)$ is the consumption rate at time t, and where U is a utility function. Yarri considered four cases defined by the availbility of a life insurance market and by the choice of the utility function. However, for all four cases, Yarri was unable to obtain any closed-form solution.

Considerable literature (Hakansson(1969), Fischer (1973) and Richard (1975), etc) has been built on Yaari's pioneering work. In particular, Leung (1994) pointed out that Yaari's model cannot have an interior solution which lasts until the maximum lifetime for the optimal consumption.

In a discrete-time environment, Hakansson (1969) examined life-cycle patterns of consumption and saving when the individual's lifetime is a discrete random variable taking values in $[0, T]$ and having a known probability distribution. The individual's objective is to maximize the expected utility from consumption as long as he lives and from the bequest left upon his death. Hakansson also briefly considered the optimal insurance purchase. Within a similar, discrete-time framework, Fischer (1973) examined life-cycle patterns of optimal insurance purchase in detail using the dynamic programming technique and obtained the formula for the present value of the future income, a formula that is different from the one under a certain lifetime. Note the maximal possible lifetime for the individual is T periods, so if the individual is alive at the beginning of the period T he will definitely be dead at the end of this period, and then the individual is inclined to buy infinite life insurance for the last period. In order to address this difficulty, Fischer imposed the artificial condition that the individual cannot buy insurance in the last period of his life. But dynamic programming for the discrete model works backward, so this artificial condition changes the whole structure of solutions.

In a second line of relevant research the emphasis was on optimal consumption and investment decisions while tending to ignore the possibility of the individual's premature death as well as the opportunity to purchase life insurance. Noteworthy in this category is the pioneering work by Merton (1969, 1971). In his model most relevant to ours the individual starts with a specified endowment and has a specified income, the planning horizon T is fixed, and the individual seeks the consumption and investment strategies that maximize the expectation of the utility of consumption plus the utility of terminal wealth. He used dynamic programming methods and obtained, in the case of constant relative risk aversion utility functions, explicit solutions.

These two lines of research converged when Richard (1975) combined Yaari's setting for an uncertain lifetime and Merton's dynamic programming approach to consider a life-cycle life insurance and consumption/investment problem with a budget constraint and objective

$$\max E\left[\int_0^{\tau} U(c(t), t)dt + B(Z(\tau), \tau)\right],$$

where τ is a *bounded* random lifetime taking values on $[0, T]$ and having a known probability distribution, U is the utility for consumption, and B is the utility for the bequest $Z(\tau)$. He also introduced the concept of a continuous-time life insurance market where the wage earner continuously buys term life insurance, the term being infinitesimally short. Richard employed the dynamic programming technique to attack this problem. To do that, Richard rewrote the objective functional in the dynamic form:

$$J(W,t) \triangleq \max E\left[\int_t^T \pi(x,t)\left[\int_t^x U(c(s),s)ds + B(Z(x),x)\right]dx\Big|\mathcal{F}_t\right], \qquad (1)$$

where $\pi(x,t)$ is the conditional probability density for death at time x conditional upon the investor being alive at time t and where W is the individual's time-t wealth. Note that Equation (1) is well-defined for $t<T$, but (1) is undefined for $t=T$ since the length of the integral interval is zero but $\pi(x,T)$ can be infinite. We know dynamic programming works backward (we can clearly see this point in the discrete-time environment), so the terminal condition is very important in dynamic programming. The assumption about the individual's behavior at $t=T$ and thus the dynamic program's boundary condition critically affects the resulting solutions.

To obtain explicit solutions for Constant Relative Risk Aversion (CRRA) utility functions, Richard imposed an artificial condition: the individual cannot purchase life insurance at time T if the individual is alive at that time. But his solution shows that the individual is inclined to purchase infinite life insurance just before his maximal possible lifetime T. That violates his imposed condition and is unrealistic, thereby compromising his results.

Subsequently, Campbell (1980), Lewis (1989) and Iwaki et al. (2004) examined the demand for life insurance from different perspectives. Campbell (1980) considered the insurance problem in a very short time interval (viz., $[t,t+\Delta t]$), used a local analysis (Taylor expansion) to greatly simplify the problem, and then derived the insurance policy in terms of the present value of the future income, the current wealth, and other parameters. Since the local analysis ignored information about the present value of the future income even if one knew the income stream, Campbell had to assume that the present value of the future income is given exogenously. Lewis (1989) examined the demand for life insurance from the perspective of beneficiaries. Iwaki et al. (2004) used martingale methods to consider the optimal insurance problem from the perspective of a household. But households are only allowed to buy life insurance at time 0 in their model, and they subsequently cannot change the amount of life insurance regardless of what happens after time 0, and so the existence of a dynamic market for life insurance is ignored. Blanchet-Scalliet, El Karouri, Jeanblanc, and Martellini (2003) considered a consumption/investment model with a stochastic time-horizon, but without life insurance and exogenous income. For CRRA utilities, they obtained explicit solutions similar to the classical Merton's results. Blanchet-Scalliet, El Karouri and Martellini (2005) addressed the problem of pricing and hedging a random cash-flow received at a random date using arbitrage argument without considering insurance.

In our paper we develop a continuous-time model that combines the best and most realistic features of all those in the literature. In particular, our model should be viewed as a natural extension of Richard (1975) since we borrowed his setting of a continuous-time life insurance market. To model the wage earner's lifetime we use the concept of uncertain life found in reliability theory; this approach is commonly used for industrial life-testing and actuarial science. In this way we overcome a critical shortcoming of Richard's (1975) work, replacing his assumption that lifetimes are bounded with the assumption that lifetimes simply take values in all of $(0, \infty)$. Moreover, we interpret the planning horizon T as the wage earner's retirement time, instead of the maximal length of his lifetime, and we introduce a utility for his wealth at this time. So our model turns out to be an intertemporal model instead of a life-cycle model. But in doing this we also assume that uncertainty about the wage earner's lifetime is independent of uncertainty about the financial market.

In section 2 we describe our model in detail, showing that it is an intertemporal model instead of a life-cycle model. There we also introduce some concepts from the survival analysis and reliability theory literatures, necessary for our modeling of the wage earner's lifetime. In section 3 we use the dynamic programming technique to obtain explicit solutions for the wage earner's optimal strategies. There it will be seen that we do not impose any artificial condition to obtain the dynamic program's terminal boundary condition; it comes naturally from the investor's objective functional. There it will also be seen that our solution differs from Richard's (1975) even though his model is a special case or ours. In section 4 we provide some numerical examples of optimal strategies, thereby providing some economic insights about basic relationships, such as how insurance payments might depend upon age and wealth. Many such findings are in accordance with *a priori* intuition, but some are not obvious and thus are thought provoking. Finally, we conclude in section 5 with a brief summary of our main results, a comparison of our results with Richard's (1975) and some thoughts about future research.

We should add that much of this paper is based upon work by Ye (2006). Moreover, Pliska and Ye (2007), which is also based upon work by Ye (2006), examined economic implications using a simplified version of this paper's model, namely, where there is no risky asset and so the only uncertainty is the wage earner's lifetime. Ye (2007) summarized a series of fundamental results which were obtained using martingale approaches in Ye (2006).

2 The Model

Let $W(t)$ be a standard 1-dimensional Brownian motion defined on a given probability space $(\Omega, \mathcal{F}, P)$. Let $T < \infty$ be a fixed planning horizon, conveniently interpreted as the wage earner's retirement time. Let $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ be the P-augmentation of the filtration $\sigma\{W(s), s \leq t\}, t \in [0,T]$, so $\mathcal{F}_t$ represents the information known to the wage earner at time t.

The continuous-time economy consists of a financial market and an insurance market. We are going to describe their details separately, beginning with the financial market.

We assume there is a risk-free security in the financial market whose time-t price is denoted by $S_0(t)$. It evolves according to

$$\frac{dS_0(t)}{S_0(t)} = r(t)dt, \tag{2}$$

where $S_0(0) = s_0$ is a given positive constant and where $r(t) : [0,T] \to R^+$ is a given continuous function describing the riskless interest rate as a function of time.

Also there is a risky security in the financial market [1] . It evolves according to the linear stochastic differential equation

$$\frac{dS_1(t)}{S_1(t)} = \mu(t)dt + \sigma(t)dW(t), \tag{3}$$

where $S_1(0) = s_1$ is a given positive constant, $\mu : [0,T] \to R$ is continuous process, and $\sigma : [0,T] \to R$ is a continuous function satisfying $\sigma^2(t) \geq k, \forall t \in [0,T]$, for some positive constant k.

Suppose the wage earner is alive at time $t = 0$ and has a lifetime denoted τ, a non-negative random variable defined on the probability space $(\Omega, \mathcal{F}, P)$. We assume the random variable τ is independent of the filtration $\mathbb{F}$ and has a probability distribution with underlying probability density function $f(t)$ and distribution function given by

$$F(t) \triangleq P(\tau < t) = \int_0^t f(u)du. \tag{4}$$

[1] Our model can be easily extended to N risky securities

Turning to the life insurance market, we assume that life insurance is available continuously and the wage earner can enter a life insurance contract by paying premiums at the rate p at each point in time. In compensation, if the wage earner dies at time t when the premium payment rate is p, then the insurance company pays an insurance amount $p/\eta(t)$. This is like term insurance where the term is infinitesimally small. Here $\eta : [0, T] \to R^+$ is a continuous, deterministic, specified function that is called *the insurance premium-payout ratio*. For example, if the wage earner dies while paying at the rate $p = 1000$ dollars per year, and if $\eta(t) = 0.01$, then the insurance company pays the estate $\$100, 000$.

Now suppose the wage earner is endowed with the initial wealth x and will receive an income at rate $i(t)$ during the period $[0, \min\{T, \tau\}]$, so the income will be terminated by the wage earner's death or retirement at T, whichever happens first. Here $i : [0, T] \to R^+$ is a given Borel measurable function that satisfies $\int_0^T i(u)du < \infty$.

The wage earner's decisions can be fully described by three processes:

$c(t) \triangleq$ Consumption rate (e.g., dollars per year) at time t, an $\{\mathcal{F}_t\}$-progressively measurable, nonnegative process

$p(t) \triangleq$ Premium rate (e.g., dollars per year) at time t, an $\{\mathcal{F}_t\}$-predictable process

$\theta(t) \triangleq$ Dollar amount in the risky security at time t, an $\{\mathcal{F}_t\}$-progressively measurable process

We let $\mathcal{A}(x)$ denote the set of all admissible 3-tuples (c, p, θ), that is, the set of all admissible decision strategies.

Given portfolio process θ, consumption process c, premium rate process p, and income process i, the wealth process $X(t)$ on $[0, \min\{T, \tau\}]$ is defined by

$$\begin{aligned} X(t) = x - \int_0^t c(s)ds - \int_0^t p(s)ds + \int_0^t i(s)ds \\ + \int_0^t \frac{X(s) - \theta(s)}{S_0(s)} dS_0(s) + \int_0^t \frac{\theta(s)}{S_1(s)} dS_1(s) \qquad \text{on} \quad t \le \min\{\tau, T\}. \end{aligned} \tag{5}$$

Using (2) and (3), we write (5) as the stochastic differential equation

$$
\begin{aligned}
dX(t) = & \, r(t)X(t)dt - c(t)dt - p(t)dt + i(t)dt \\
& +\theta(t)[(\mu(t) - r(t))dt + \sigma(t)dW(t)] \qquad \text{on} \quad t \leq \min\{\tau, T\}.
\end{aligned} \tag{6}
$$

If the wage earner dies at time t, $0 < t \leq T$, the estate will get the insurance amount $p(t)/\eta(t)$. Then the wage earner's total legacy when he/she dies at time t with wealth $X(t)$, that is, the total value of the estate, is

$$
Z(t) = X(t) + \frac{p(t)}{\eta(t)} \qquad \text{on} \quad \{\tau = t\}. \tag{7}
$$

The wage earner's problem is to choose life insurance purchase, consumption, and portfolio investment strategies so as to maximize the expectation of utility from three sources: from consumption during $[0, min\{\tau, T\}]$, from the legacy if he/she dies before time T, and from the terminal wealth if he/she is alive at time T. We assume the wage earner's preferences are based upon discounted Constant Relative Risk Aversion (CRRA) utility functions. Thus the maximum expected utility is expressed as

$$
\begin{aligned}
V(x) \triangleq \sup_{(c,p,\theta)\in\mathcal{A}(x)} E\Bigg[& \int_0^{T\wedge\tau} \frac{e^{-\rho s}}{\gamma}(c(s))^\gamma ds + \frac{e^{-\rho\tau}}{\gamma}(Z(\tau))^\gamma 1_{\{\tau\leq T\}} \\
& + \frac{e^{-\rho T}}{\gamma}(X(T))^\gamma 1_{\{\tau > T\}}\Bigg],
\end{aligned} \tag{8}
$$

where $T \wedge \tau \triangleq \min\{T, \tau\}$. We assume that $\gamma < 1$ and $\rho > 0$. It is well accepted that the utility functions are logarithmic when $\gamma = 0$.

We show how to solve this problem in the following section. To do so we shall model the wage earner's lifetime by making important use of some concepts from the reliability theory and survival analysis literatures (see, e.g., Collett, 2003), so it is appropriate to conclude this section by introducing some relevant notation.

The function $\bar{F}(t)$, which is called the *survivor function*, is defined to be the probability that the lifetime is greater than or equal to t, i.e.,

$$
\bar{F}(t) \triangleq P(\tau \geq t) = 1 - F(t). \tag{9}
$$

The *hazard function* represents the instantaneous death rate for the wage earner who has survived to time t, and it is defined by

$$
\lambda(t) \triangleq \lim_{\delta t\to 0} \frac{P(t \leq \tau < t + \delta t \mid \tau \geq t)}{\delta t} = \frac{f(t)}{\bar{F}(t)}, \tag{10}
$$

where the last equality follows from definitions and basic relationships. From this it follows that

$$\lambda(t) = -\frac{d}{dt}(\ln \bar{F}(t)),$$

in which case the survivor function is given by

$$\bar{F}(t) = \exp\left\{ -\int_0^t \lambda(u)du \right\} \tag{11}$$

and the probability density function is related to the hazard rate by

$$f(t) = \lambda(t)\exp\left\{ -\int_0^t \lambda(u)du \right\}. \tag{12}$$

From these equations we see there is a correspondence between hazard functions and density functions. Hence throughout the rest of this paper we suppose that *the hazard function* $\lambda(t)$ *is given* and $\lambda(t) : [0,\infty] \to R^+$ is a continuous, deterministic function which satisfies $\int_0^\infty \lambda(t)dt = \infty$. Then the probability density of τ is given by (12) and the survivor function is given by (11).

We now introduce some additional notation associated with the random variable τ. Denote by $f(s,t)$ the conditional probability density for death at time s conditional upon the wage earner being alive at time $t \leq s$, so that

$$f(s,t) \triangleq \frac{f(s)}{\bar{F}(t)} = \lambda(s)\exp\left\{ -\int_t^s \lambda(u)du \right\}. \tag{13}$$

And denote by $\bar{F}(s,t)$ the conditional probability for the wage earner being alive at time s conditional upon being alive at time $t \leq s$, so that

$$\bar{F}(s,t) \triangleq \frac{\bar{F}(s)}{\bar{F}(t)} = \exp\left\{ -\int_t^s \lambda(u)du \right\}. \tag{14}$$

3 Stochastic Dynamic Programming

In this section we shall use stochastic dynamic programming methods to derive explicit formulas for the wage earner's optimal decision strategies. In particular, we shall use the optimality principle to set up the Hamilton-Jacobi-Bellman (HJB) equation, and then we shall solve this equation for the optimal strategies.

To derive the HJB equation we need to restate (8) in a dynamic programming form. To begin, for any (c, p, θ) define the corresponding expected utility, starting with wealth x at time t:

$$J(t,x;c,p,\theta) \triangleq E\Big[\int_t^{T\wedge\tau} \frac{e^{-\rho s}}{\gamma}(c(s))^{\gamma}ds + \frac{e^{-\rho\tau}}{\gamma}(Z(\tau))^{\gamma}1_{\{\tau\leq T\}} + \frac{e^{-\rho T}}{\gamma}(X(T))^{\gamma}1_{\{\tau>T\}}\Big]. \tag{15}$$

The next step is to transform our problem having a random planning horizon to an equivalent problem having a fixed planning horizon, thereby permitting us to proceed with the HJB equation. Since τ is independent of the filtration $\mathbb{F}$, we can achieve the necessary transformation (refer to Ye (2006)) by using an equivalent form of $J(t, x; c, p, \theta)$:

$$J(t,x;c,p,\theta) = E\Big[\int_t^{T} [f(u,t)\frac{e^{-\rho u}}{\gamma}(Z(u))^{\gamma} + \bar{F}(u,t)\frac{e^{-\rho u}}{\gamma}(c(u))^{\gamma}]du + \bar{F}(T,t)\frac{e^{-\rho T}}{\gamma}(X(T))^{\gamma}\Big|\mathcal{F}_t\Big], \tag{16}$$

where $f(u, t)$ is given by (13) and $\bar{F}(u, t)$ is given by (14).

From this result we see that the wage earner who faces unpredictable death acts as if he or she will live until time T, but with a subjective rate of time preferences equal to his or her "force of mortality" for his/her consumption and terminal wealth. From the mathematical point of view, this result enables us to state the dynamic programming principle. This is a recursive relationship for the maximum expected utility as a function of the wage earner's age and his/her wealth at that time:

$$V(t,x) = \sup_{(c,p,\theta)\in\mathcal{A}(t,x)} E\Big[\exp\Big\{-\int_t^{s}\lambda(v)dv\Big\}V(s,X(s)) + \int_t^{s} f(u,t)\frac{e^{-\rho u}}{\gamma}(Z(u))^{\gamma} + \bar{F}(u,t)\frac{e^{-\rho u}}{\gamma}(c(u))^{\gamma}du\Big|\mathcal{F}_t\Big].$$

Here the definition of the set of admissible strategies $\mathcal{A}(t, x)$ is similar to the definition $\mathcal{A}(x)$, except that the starting time is time t and the wealth at time t is x.

We are now in a position to present the dynamic programming equation, that is, the HJB equation:

$$\begin{cases} V_t(t,x) - \lambda(t)V(t,x) + \sup_{(c,p,\theta)} \Psi(t,x;c,p,\theta) = 0 \\ V(T,x) = \frac{e^{-\rho T}}{\gamma} x^\gamma, \end{cases} \tag{17}$$

where

$$\begin{aligned}\Psi(t,x;c,p,\theta) \triangleq\, & (r(t)x + \theta(\mu(t) - r(t)) + i(t) - c - p)V_x(t,x) \\ & + \frac{1}{2}\sigma^2(t)\theta^2 V_{xx}(t,x) + \lambda(t)\frac{e^{-\rho t}}{\gamma}(x + p/\eta(t))^\gamma + \frac{e^{-\rho t}}{\gamma}c^\gamma.\end{aligned}$$

We refer to Ye (2006) for deriving this HJB equation. We should point out that the PDE part of this HJB equation is the same as Richard (1975). The boundary condition is naturally obtained by setting t to T in (16), while Richard artificially set the boundary condition to 0 since he cannot obtain the boundary condition from his model. This HJB equation enables us to derive the optimal insurance, portfolio and consumption strategies. According to Ye (2006), if V is a solution of the HJB equation (17) and if an admissible 3-tuple (c^*, p^*, θ^*) satisfies

$$\begin{aligned} 0 &= V_t(t,x) - \lambda(t)V(t,x) + \Psi(t,x;c^*,p^*,\theta^*) \\ &= V_t(t,x) - \lambda(t)V(t,x) + \sup_{(c,p,\theta)} \Psi(t,x;c,p,\theta), \end{aligned} \tag{18}$$

then V is the maximum expected utility function and (c^*, p^*, θ^*) are the optimal strategies. To exploit this, we first use the first-order conditions for a regular interior maximum to (18) in order to derive the following expressions for the optimal strategies in terms of the solution V:

$$c^*(t) = \left[\frac{1}{V_x e^{\rho t}}\right]^{1/(1-\gamma)}, \tag{19}$$

$$x + \frac{p^*(t)}{\eta(t)} = \left(\frac{1}{V_x e^{\rho t}}\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)}, \tag{20}$$

and

$$\theta^* = -\frac{(\mu(t) - r(t))V_x}{\sigma^2(t)V_{xx}}. \tag{21}$$

We now plug (19)-(21) in (17) and combine the similar terms to get

$$-\lambda(t)V + V_t - \frac{1}{2}\left(\frac{\mu(t)-r(t)}{\sigma(t)}\right)^2 \frac{V_x^2}{V_{xx}} + [(r(t)+\eta(t))x + i(t)]V_x$$
$$+\frac{1-\gamma}{\gamma}e^{-\rho t/(1-\gamma)}\left[\frac{(\lambda(t))^{1/(1-\gamma)}}{(\eta(t))^{\gamma/(1-\gamma)}}+1\right]V_x^{-\gamma/(1-\gamma)} = 0. \tag{22}$$

It thus remains to determine the solution V of differential equation (22) subject to the boundary condition in (17). The ensuing analysis is rather lengthy and technical (see Ye (2006) for the details), so we proceed right to the solution:

$$V(t,x) = \frac{a(t)}{\gamma}(x+b(t))^{\gamma}, \tag{23}$$

where

$$a(t) = e^{-\rho t}(e(t))^{1-\gamma}, \tag{24}$$

$$b(t) = \int_t^T i(s)\exp\left\{-\int_t^s [r(v)+\eta(v)]dv\right\} ds, \tag{25}$$

$$e(t) = \exp\left\{-\int_t^T H(v)dv\right\} + \int_t^T \exp\left\{-\int_t^s H(v)dv\right\} K(s)ds, \tag{26}$$

$$H(t) \triangleq \frac{\lambda(t)+\rho}{1-\gamma} - \frac{1}{2}\gamma\left(\frac{\mu(t)-r(t)}{(1-\gamma)\sigma(t)}\right)^2 - \frac{\gamma}{1-\gamma}(r(t)+\eta(t)), \tag{27}$$

and

$$K(t) \triangleq \frac{(\lambda(t))^{1/(1-\gamma)}}{(\eta(t))^{\gamma/(1-\gamma)}} + 1. \tag{28}$$

Consequently, from (19)-(21), (23) and (24) the optimal consumption, bequest, and portfolio strategies can be explicitly written as

$$c^*(t) = \frac{1}{e(t)}(x+b(t)), \tag{29}$$

$$Z^*(t) = x + \frac{p^*(t)}{\eta(t)} = \left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} \frac{1}{e(t)}(x+b(t)), \tag{30}$$

and

$$\theta^*(t) = \frac{\mu(t) - r(t)}{(1-\gamma)\sigma^2}(x + b(t)). \tag{31}$$

From (30), the optimal strategy for insurance premium payments is given by

$$p^*(t) = \eta(t)\left\{\left[\left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} \frac{1}{e(t)} - 1\right] x + \left(\frac{\lambda(t)}{\eta(t)}\right)^{1/(1-\gamma)} \frac{b(t)}{e(t)}\right\}. \tag{32}$$

Notice that all these optimal strategies have been presented in feedback form, depending not only on the current time t but also on the time-t wealth x.

It is worth remarking that the function $b(t)$ represents the fair value at time t of the wage earner's future income from time t to time T. In other words, this is the maximum amount a bank will lend the individual in a riskless loan, hedging the risk of premature death by purchasing a suitable amount of life insurance at the borrower's expense. Because of its interpretation and importance, we call $b(t)$ "the value of human capital." The formula for $b(t)$ were derived with arbitrage arguments by Ye (2006).

The sum of the current wealth and the value of human capital, viz., $x + b(t)$, is a quantity that stands out in all three of the formulas (29)-(31) for the optimal strategies. Because of its apparent importance we shall refer to this sum as "the wage earner's total overall wealth." Note from (29) that the optimal consumption as a *fraction* of the total overall wealth is simply equal to the deterministic function of time $e(t)^{-1}$. And the same can be said about the optimal bequest Z if the payout ratio η is a simple multiple of the hazard rate λ. Finally, note that if the risky asset's appreciation rate μ and the riskless interest rate r are constants, then the optimal *fraction* of the overall wealth that should be invested in the risky asset is a constant, independent of age. This last result is entirely consistent with well known results for classical portfolio management problems (see Merton (1969, 1971)) in the case of CRRA utility functions.

4 Numerical Results and Economic Implications

In this section we use explicit solutions of a numerical example to examine economic implications of our model. Throughout we consider a wage earner who starts work at age 25, is planning to retire at age 65, and whose initial wage at age 25 is $\$50,000$, growing at the rate 3% per year. Other parameters are given in Table 1.

Table 1
The Parameters

Para.	T	r	μ	σ	ρ	$\lambda(t)$	$\eta(t)$	γ
Val.	40	0.04	0.09	0.18	0.03	$.001 + \mathrm{e}^{-9.5+.1t}$	$1.05\lambda(t)$	-3

Since it plays such an important role in our results, we show in Figure 1 a graph of $b(t)$, the arbitrage value of human capital for this example.

In the remaining figures we varied one and only one parameter each time in order to examine each parameter's effect on optimal insurance purchase decisions. For each parameter we produce the new optimal life insurance payment function and then subtract the baseline optimal life insurance payment function to produce Figures 2–8.

For Figure 2 we doubled the riskless interest rate from 4% to 8%. The wage earner then buys less life insurance, probably because the riskless investment has become more attractive.

For Figure 3 we doubled the discount rate ρ from 3% to 6%. This makes little difference when the wealth is small, but wealthy persons nearing retirement should purchase more life insurance, that is, they should not sell as much.

For Figure 4 we changed the risk aversion parameter γ from -3 to -1, thereby making the wage earner less risk averse. This has little effect until the wage earner nears retirement, at which time he or she should spend less on insurance, i.e., should sell more insurance.

For Figure 5 we doubled the hazard rate λ, so the wage earner's life expectancy is considerably reduced. For the most part there is little difference in the amount of insurance purchased, but the less healthy wage earner nearing retirement will want to buy more life insurance than the healthy one.

For Figure 6 we doubled the insurance premium-payout ratio η from 1.05 to 1.1 times λ, thereby making the insurance policy more expensive for the wage earner. We see this change has an undetermined effect on the wage earner's life insurance purchase. For small levels of wealth the wage earner faced with a more expensive policy will "bite the bullet" and spend more on insurance, but for larger levels of wealth this wage earner will actually spend less.

For Figure 7 we doubled the risky security's expected rate of return μ from 9% to 18%. We see the wage earner with the better investment opportunity will buy more life insurance, especially when nearing retirement with a large wealth. Perhaps this is because this wage earner finds it unnecessary to save as much in order to achieve similar investment objectives.

Finally, for Figure 8 we doubled the volatility σ of the risky security from 18%

to 36%. We see the wage earner with the more volatile investment opportunity will buy less life insurance, especially for wage earners in their middle years. Perhaps this is because this wage earner will find it necessary to save more in order to achieve similar investment objectives.

5 Discussion

In this paper we have developed a comprehensive, continuous-time model of lifetime optimal consumption, optimal investment, and optimal life insurance purchase for a wage earner. Our model can be viewed as a natural extension of Merton's (1969), (1971) model of optimal consumption and investment for a wage earner having a specified income, the key extensions being the random lifetime of the wage earner, the opportunity to purchase life insurance, and the utility of the bequest upon the wage earner's death before retirement. By assuming the wage earner has CRRA utility functions we were able to derive explicit formulas for the optimal strategies. We then used these formulas in conjunction with a numerical example to study the economic implications of our results.

Our model clearly resembles that of Richard (1975), to the extent that the stochastic differential equations for his and our wealth processes are the same and even the PDE for his and our dynamic programs are the same. However, there are some important differences. His wage earner has a bounded life time, and the planning horizon is the least upper bound on this lifetime. In our case the wage earner's lifetime is not necessarily bounded, and the planning horizon is explained as the wage earner's planned time of retirement, with a utility of his wealth at that time. Looking at his and our explicit solutions, one might be tempted to conclude that they coincide, since they have exactly the same form (see (23)). However, this is not the case, because his function $a(t)$ is different from ours. In fact, he has $a(T) = 0$ for T equal to his planning horizon, where our function satisfies $a(t) > 0$ for all t.

For the most part the implications of our analysis are in accordance with economic intuition. For example, the optimal expenditure for insurance is a decreasing function of the wage earner's overall wealth and is a unimodal function of age, reaching a maximum at an intermediate age. On the other hand, our model has revealed some relationships whose economic explanations are unclear, such as why an increase in the discount rate ρ would cause the wage earner to purchase more life insurance.

A perhaps controversial aspect of our model is the fact that optimal solutions can call for the wage earner to *sell* a life insurance policy on his or her own life. This is important and informative, because it certainly suggests practical circumstances when a person should not purchase any life insurance, but it is hardly realistic. To deal with this it is necessary to impose a constraint requiring the insurance purchase rate $p(t)$ to be nonnegative. But doing so seriously damages the hopes of obtaining explicit solutions, making it necessary to use numerical methods to solve the corresponding HJB equation. Some numerical work in this direction has been carried out by Ye (2006).

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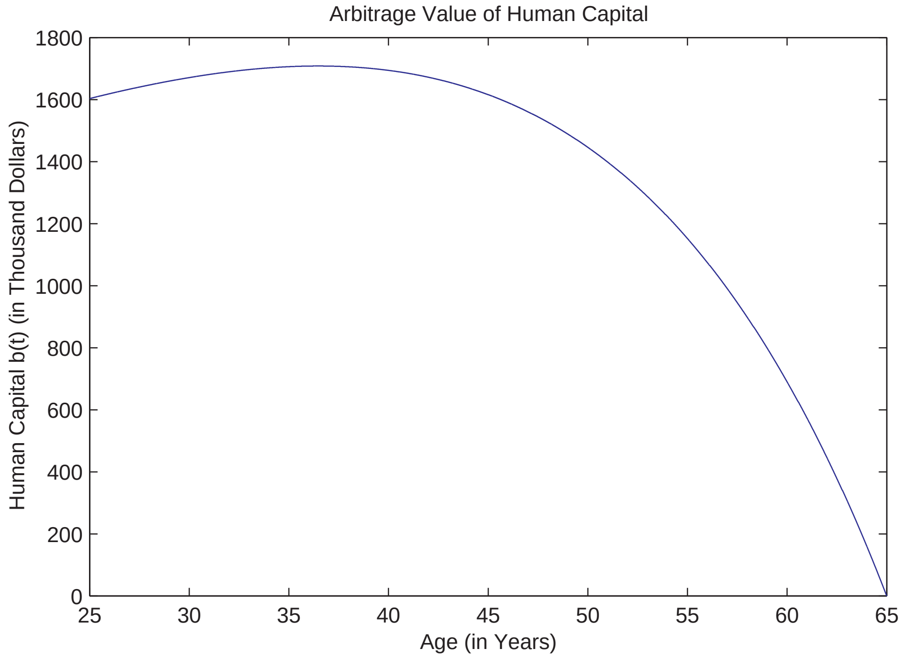

Fig. 1. The fair value of the human capital

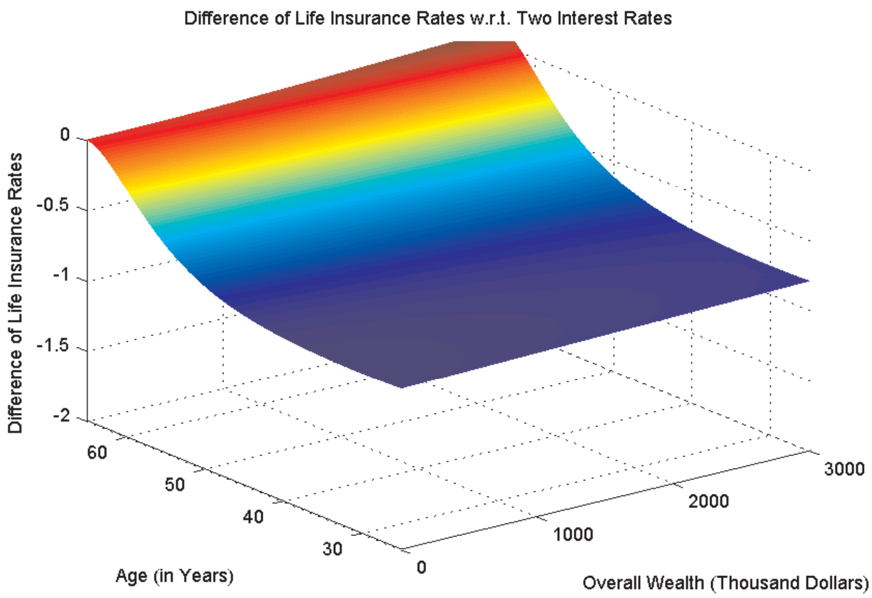

Fig. 2. The difference of life insurance rates between the case of twice the base interest rate and the case of the base interest rate

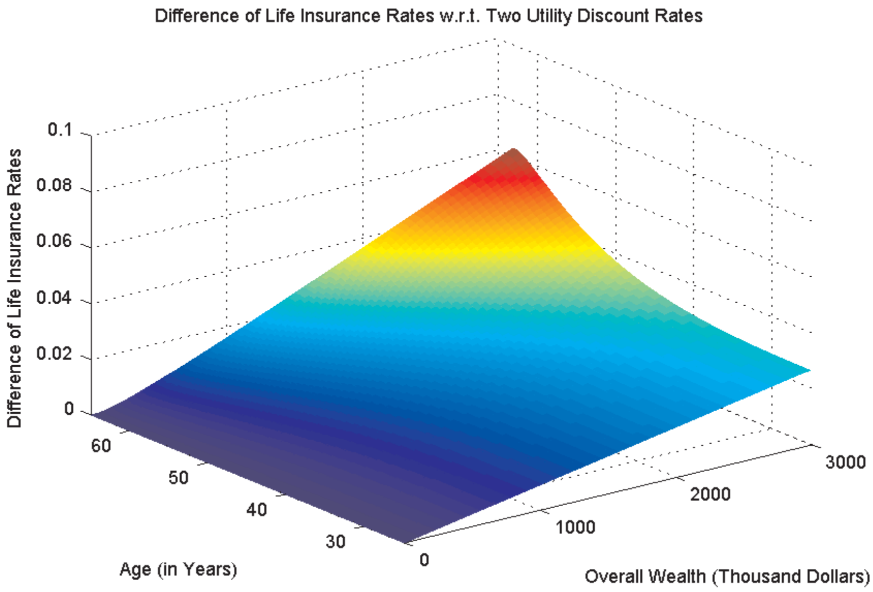

Fig. 3. The difference of life insurance rates between the case of twice the base utility discounted rate and the case of the base utility discounted rate

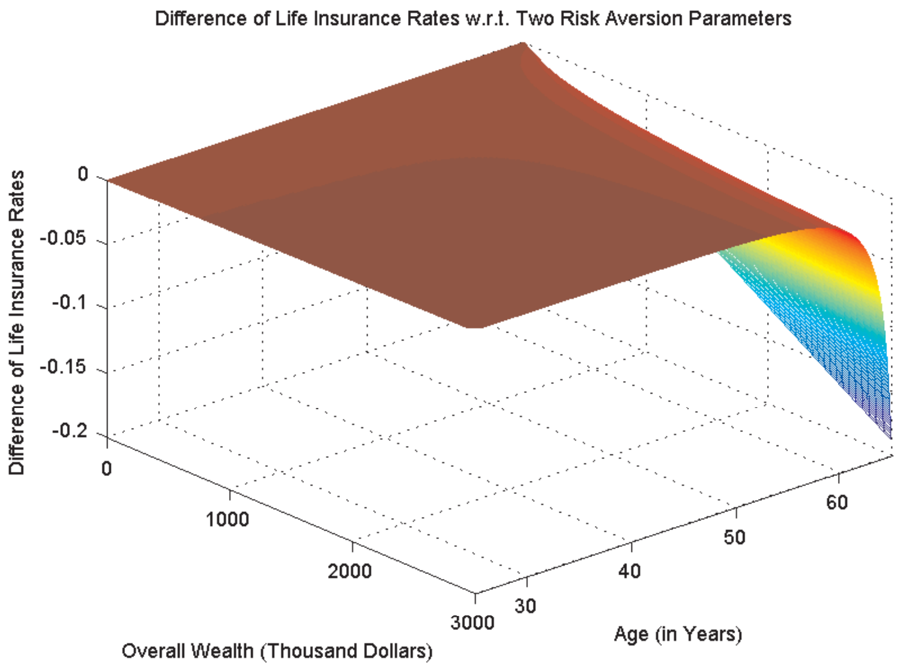

Fig. 4. The difference of life insurance rates between the risk aversion parameter $\gamma = -1$ and the case of the base risk aversion parameter

Fig. 5. The difference of life insurance rates between the case of twice the base hazard rate and the case of the base hazard rate

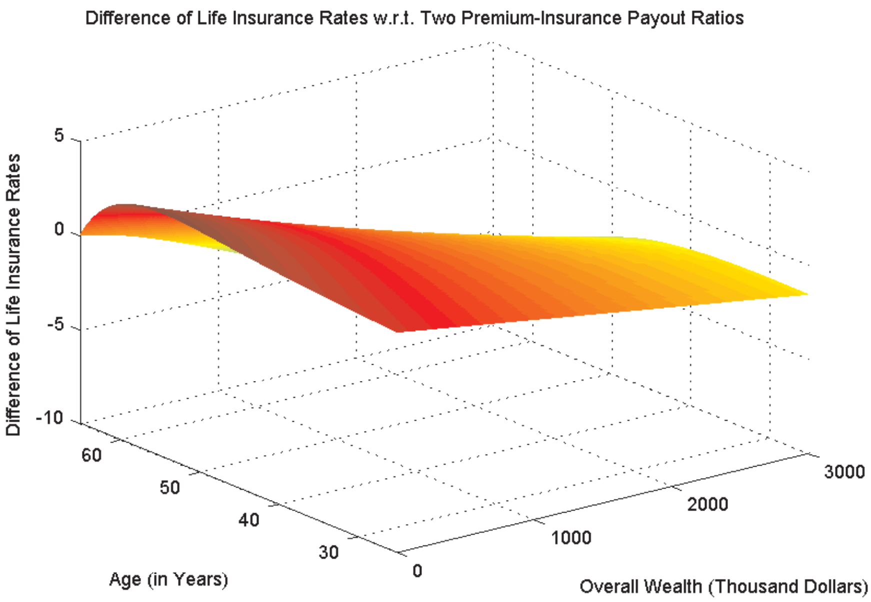

Fig. 6. The difference of life insurance rates between the case of twice the base insurance premium - payout ratio and the case of the base insurance premium - payout ratio

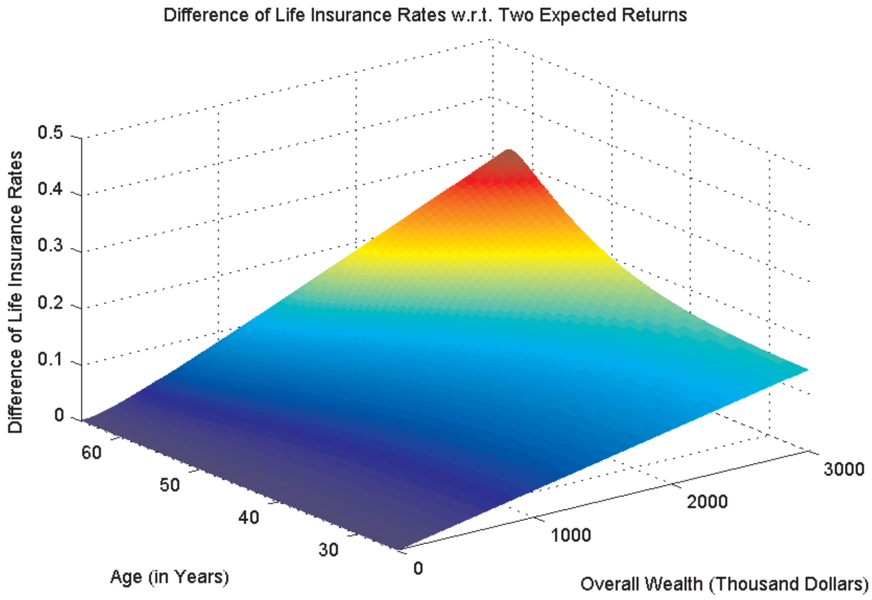

Fig. 7. The difference of life insurance rates between the case of twice the base expected return of the risky security and the case of the base expected return of the risky security

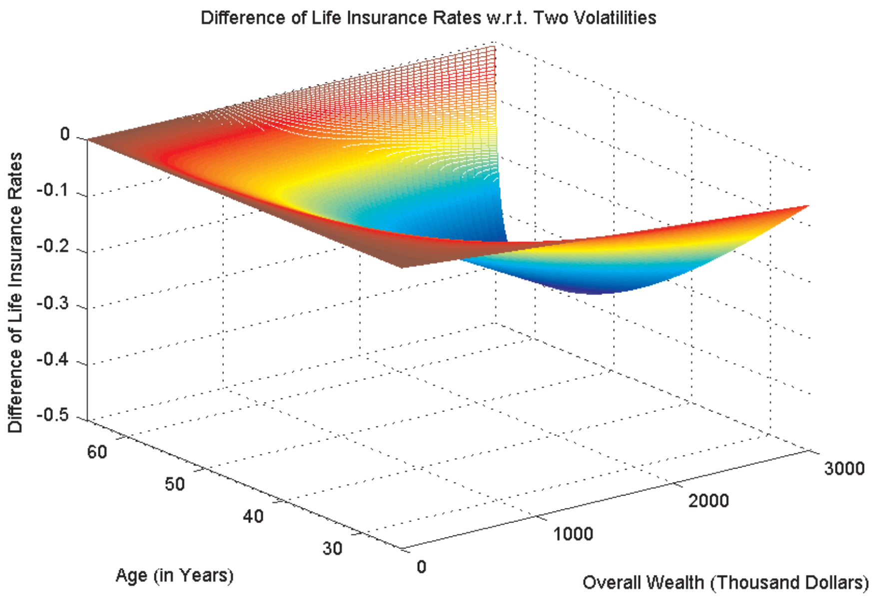

Fig. 8. The difference of life insurance rates between the case of twice the base volatility of the risky security and the case of the base volatility of the risky security