A 3-factor Valuation Model for Mortgage-Backed Securities (MBS)

TAKEAKI KARIYA
Research Center for Financial Engineering
KIER, Kyoto University
Sakyo-ku, Kyoto 606-8501 Japan
kariya@kier.kyoto-u.ac.jp

FUMIAKI USHIYAMA
Department of Social Informatics
Kyoto University
Sakyo-ku, Kyoto 606-8501, Japan
ushiyama@kuis.kyoto-u.ac.jp

and

STANLEY R. PLISKA*
Department of Finance
University of Illinois at Chicago
601 S. Morgan Street, Chicago, IL 60607-7124 USA
srpliska@uic.edu

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Abstract

In this paper we generalize the one-factor MBS-pricing model proposed by Kariya and Kobayashi(2000) to a 3-factor model. We describe prepayment behavior due to refinancing and rising housing prices by incentive response functions. Our valuation of an MBS is based on discrete-time, no-arbitrage pricing theory, making an association between prepayment behavior and cash flow patterns. The structure, rationality, and potential for practical use of our model is demonstrated by valuing an MBS via Monte Carlo simulation and then conducting a comparative statics analysis.

1 Introduction

Via a no-arbitrage pricing theory in a discrete time setting, Kariya and Kobayashi(2000) (abbreviated KK(2000) or simply KK below) formulated a framework for pricing an MBS (Mortgage-Backed Security) and proposed

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a one-factor valuation model that has the capacity to describe the burnout-effect of prepayment. The framework directly embeds the heterogeneity of prepayment behavior into the valuation of an MBS. A special feature of their framework is the treatment of the prepayment option given to loan borrowers (mortgagors) for valuing an MBS. Their approach is important because, in the prevailing literature, as represented by Stanton (1995), when a theoretical valuation is attempted, the value of the prepayment option is regarded as a gross or lump-sum value, and the value of the MBS is decomposed into this option part and the value of a riskless bond. However, this option part is usually based upon a representative mortgagor about whom very particular assumptions are made with respect to his or her decision making behavior. In particular, this option-based approach implicitly and usually assumes homogeneous mortgagors. But, in fact, the mortgagors in an MBS pool are typically heterogeneous, and this heterogeneity affects the distribution of prepayments and hence the value of an MBS.

The KK framework provides an association between the pattern of cash flows and the heterogeneous prepayment behavior of the mortgagors. They only treated prepayment due to refinancing and expressed the heterogeneity of prepayment behavior in terms of different incentive thresholds for changes of mortgage interest rates. In doing so they employed a one-factor model and assumed that the mortgage rate is a linear function of a short-term interest rate that discounts the cash flow of the MBS to a present value. But this linkage of the two interest rates has a serious shortcoming, as was pointed out in that paper. For instance, a big decrease in the mortgage rate, which typically accompanies a decrease in the short-term rate, will in general tend to lower the value of an MBS due to the refinancing, whereas a decrease in the short-term rate tends to increase the value of an MBS through increasing the discount factors. Therefore it is important to distinguish these two interest rates and let them play their separate roles.

In this paper, we extend the KK model in the following two ways:

(1) there is a distinction between the short-term interest rate used for discounting and the mortgage rate used as an incentive factor for refinancing, and

(2) there is a second prepayment incentive factor that is based upon rising property values.

The second point (2) is clearly important, especially in valuing U.S. MBS's, because a significant increase in equity value often causes the sale of a house in order to withdraw equity. Actually, this same incentive can cause a home owner to refinance the mortgage, even if interest rates do not decline. But to reduce confusion, we refer to the first incentive factor, which is due to declining mortgage rates, as the "refinancing" factor, whereas the second incentive factor, which is related to rising housing prices, will be referred to as the "equity" factor.

We should point out that the sale of houses, and thus prepayment, is also caused by noneconomic or demographic reasons such as death of an owner or a spouse, change of job, etc. These are reasons that are essentially independent of interest rates and property values. In our present model these exogenous causes of prepayment are not included, though our framework is such that they can readily be included as part of a future research project.

In this paper, the heterogeneity of prepayment behavior is treated as that of incentive thresholds for changes of mortgage rates and property values, with different mortgagors having different thresholds. Of course, the differences between the thresholds reflect different prepayment costs, wealth levels, and so forth. Our analytical approach to the treatment of this heterogeneity of prepayments is closer to that for credit risk analysis than that with the option-based approach. In our model, a loan borrower will prepay only if a change in either his house price or mortgage rate goes over his corresponding threshold for equity or refinancing.

The distribution of the thresholds for each individual mortgagor in a loan pool is assumed to be a bivariate normal distribution. Once one of the two variables, viz., mortgage rate and house price, which are modeled by stochastic processes, hits a corresponding threshold in a 2-dimensional region, a prepayment occurs and the cash flow pattern changes, affecting the value of the MBS. Thus our valuation structure is symbolically expressed as

$$(\{r_n\},\ \{R_n\},\ \{P_n\},\ N(\boldsymbol{\mu}, \boldsymbol{\Sigma}))\,, \tag{1.1}$$

where $\{r_n\}$ and $\{R_n\}$ are respectively short-term interest rate and mortgage rate processes, $\{P_n\}$ is the house price process, and $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is a bivariate normal distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ for the distribution of thresholds for each mortgagor in a loan pool. In particular, we model the house price process by a discrete time diffusion model with an exponentially smoothing drift model. Hence the house price process is non-Markovian, which is realistic and is allowed because of our discrete time no-arbitrage approach, requiring only that discounted prices are martingales under a risk-neutral probability measure.

The bivariate normal distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ describes the heterogeneity of thresholds for prepayments and provides the boundaries that the two incentive factors $\{R_n\}$ and $\{P_n\}$ may hit, while the short-term rate process $\{r_n\}$ provides the discount factors. Thus the three-factor structure (1.1) generates prepayments in a loan pool and hence a pattern of cash flows from the MBS. Therefore the value of a given MBS can be evaluated through the structure as a forward looking value via the no-arbitrage theory.

There is a large body of literature on U.S. MBS's, both theoretical and empirical. Among others, Schwartz and Torous(1989) empirically model prepayment or defaults as a function of some explanatory variables. An important issue in a theoretical treatment of prepayment for valuation of an MBS is how option theory is applied in describing heterogeneous prepayment behavior. Examples of option-based prepayment models include Dunn and McConnell(1981a,b), Timmis(1985), Dunn and Spatt(1986) and Johnston and Van Drunen(1988). Though cost and lag are introduced as frictional factors in some of these articles, homogeneous prepayment behavior are basically treated. Stanton(1995) proposed a comprehensive prepayment model that associates heterogeneous behavior with prepayment cost. While these models have attractive features and do a reasonable job of explaining actual prepayments, they assume interest rates are the only source of risk. In Kau, Keenan, Muller and Epperson(1992,1995), Kau and Keenan(1995), and Deng, Quigley and Van Order(2000), default factor is added to the interest rate factor in their option based models, though they recognize the importance of the role of house price as a determinant of mortgage termination. Except for the last paper, they assume homogeneous prepayment behavior.

Recently Downing, Stanton and Wallace(2001) developed an option based model that handles both prepayment and default and allow for a direct impact of house prices on mortgage termination. They find that allowing house prices to affect prepayment directly allows the model to describe observed termination behavior significantly.

A common feature of these approaches is that they treat option based models in a continuous time setting, but no association is made between the cash flow pattern of an MBS and the time distribution of occurrences of prepayments. In other words, options given to mortgagors are separated from the cash flow pattern that changes according to specific occurrences of prepayments in time series and are valued separately from the pattern of changing cash flows. On the other hand, the discrete time KK(2000) approach of directly embedding prepayment behavior into the cash flow pattern for valuation of an MBS is extended to continuous time by Nakamura(2001), who obtains a semi-analytic valuation formula for an MBS.

The plan for this paper is as follows.

2 Cash Flows from an MBS with Prepayment

In this section we describe the cash flow from an MBS with prepayment, where defaults are protected by a guaranty institution. We only consider an MBS based on fixed rate loans with equal monthly payment. Let R_n be the mortgage rate at month n, C the coupon of the MBS and S the servicing rate including the guarantee. All these rates are annual rates. Also let N be the maturity month, m the current month for valuing the MBS for the remaining periods when the prepayment history up to m is given, and n

a future month ($0 \leq m \leq n \leq N$). Also let MB_n denote the remaining principal balance at the end of month n when no prepayment occurs. Then, as is well known, the constant monthly payment is

$$MP = MB_0 \times \frac{R_0/12(1+R_0/12)^N}{(1+R_0/12)^N - 1}, \tag{2.1}$$

the initially scheduled interest payment for month n is

$$I_n = MB_{n-1} \times \frac{R_0}{12}, \tag{2.2}$$

and the remaining balance at n with no prepayment allowed is

$$MB_n = MB_0 \times \frac{(1+R_0/12)^N - (1+R_0/12)^n}{(1+R_0/12)^N - 1} \quad (n = 1, \ldots, N). \tag{2.3}$$

Next, let $\overline{MB}_n$ denote the actual principal balance at the end of month n when prepayments can occur and let $\overline{I}_n$ denote the unscheduled interest paid at n under prepayment. To relate actual cash flows with prepayment structure, we assume that there are K loan borrowers in the pool and the loan sizes are equal, where K, it will turn out, is only a latent variable used to describe proportions of prepayments in terms of the number of borrowers in the pool who prepay. It is noted that this assumption enables us to switch the concept of a prepayment ratio measured in terms of money into the concept of a prepayment ratio that is measured in terms of the number of remaining borrowers. It is also assumed that there is no partial prepayment.

Denote the random variable

$$L_n = \text{the number of borrowers who prepay up to } n. \tag{2.4}$$

Then the actual principal balance for month n is expressed in terms of L_n and K as

$$\overline{MB}_n = \frac{MB_n}{K} \times (K - L_n) = MB_n \left(1 - \frac{L_n}{K}\right), \tag{2.5}$$

and the actual interest paid to investors at n is

$$\begin{aligned} \overline{I}_n &= \overline{MB}_{n-1} \times \frac{R_0}{12} = MB_{n-1}\left(1 - \frac{L_{n-1}}{K}\right)\frac{R_0}{12} \\ &= I_n \times \left(1 - \frac{L_{n-1}}{K}\right). \end{aligned} \tag{2.6}$$

Using these definitions, the total cash flow at n from the MBS is the change of the actual principal balance from $n-1$ to n and the actual interest paid at n with the servicing fee deducted:

(2.7)

$$\begin{aligned}\overline{CF}_n &= (\overline{MB}_{n-1} - \overline{MB}_n) + \frac{C}{C+S}\overline{I}_n \\ &= MB_{n-1}\left(1 - \frac{L_{n-1}}{K}\right) - MB_n\left(1 - \frac{L_n}{K}\right) + \frac{C}{C+S}I_n\left(1 - \frac{L_{n-1}}{K}\right) \\ &= a_n\left(1 - \frac{L_n}{K}\right) + b_n\left(1 - \frac{L_{n-1}}{K}\right),\end{aligned}$$

where

$$\begin{aligned}a_n &:= -MB_n \\ b_n &:= MB_{n-1} + \frac{C}{C+S}I_n.\end{aligned}$$

Note that a_n and b_n are known at 0 and hence the unknown variables in (2.7) are only the random prepayment proportions L_n/K and L_{n-1}/K. It should be noted again that the prepayment proportions are switched from the concept of the remaining principal balances to that of the remaining borrowers. Thus we can relate the actual cash flows to prepayment activities of borrowers in the pool.

Now as in KK(2000), by a general no-arbitrage pricing theory for a discrete time framework, the no-arbitrage value at time m of the MBS with maturity N is given by

$$V(m,N) = \sum_{n=m+1}^{N} CF(m,n), \tag{2.8}$$

where

$$\begin{aligned}CF(m,n) &:= E^*_m[\Delta(m,n)CF_n] \\ &= E^*_m\left[\Delta(m,n)\left\{a_n\left(1 - \frac{L_n}{K}\right) + b_n\left(1 - \frac{L_{n-1}}{K}\right)\right\}\right]\end{aligned} \tag{2.9}$$

with

$$\Delta(m,n) := \exp\left(-\sum_{j=m}^{n-1} r_j h\right) \quad (h = 1/12). \tag{2.10}$$

Here $\{r_j\}$ is the short-term riskless interest rate process and $E^*_m[\cdot]$ is the conditional expectation at m under a risk neutral measure for $\{r_j\}$, $\{R_j\}$ and $\{P_j\}$. Since the model is incomplete, we take an actual measure as such a risk neutral measure, which still guarantees no-arbitrage valuation. Note that $\Delta(m,n)$ randomly discounts a cash flow at n to a value at m.

3 Three-factor model

In KK(2000), it was assumed that a borrower in a pool prepays at n when the difference between the initial mortgage rate R_0 and the current rate R_n first exceeds his incentive threshold for the first time. Thus the condition for prepayment by the k-th borrower was formulated as

$$u_n^{(1)} := R_0 - R_n \geq d_k^{(1)}. \tag{3.1}$$

Moreover, KK(2000) assumed that the mortgage rate R_n and spot rate r_n are linearly related to each other, so the process $\{r_n\}$ determined both prepayment due to refinancing via (3.1) and the discount factor in (2.10).

In the extended model of this paper we directly use mortgage rates $\{R_n\}$ to describe prepayment behavior for refinancing as in (3.1). The threshold $d_k^{(1)}$ in (3.1) in general depends on n and some other state variables such as the loan-to-value ratio and business conditions, but for simplicity in this paper we assume that it is constant over time. Meanwhile, a separate short-term spot rate process $\{r_n\}$ plays determines the discount factor (2.10) for cash flows in (2.9). Naturally $\{R_n\}$ and $\{r_n\}$ need to be specified as processes that are highly correlated.

In addition, we assume that prepayment behavior due to the equity factor and rising property values is described in terms of economic incentives in a fashion that is similar to (3.1). More specifically, let P_n be the housing price level at n. Here it is assumed that it is a common price level for the houses in the pool, and that the k-th borrower sells his mortgaged house if the difference of the current log-price and the initial log-price exceeds or equals his threshold for equity for the first time, giving the condition

$$u_n^{(2)} = \log P_n - \log P_0 \geq d_k^{(2)}. \tag{3.2}$$

In general it is appropriate for $d_k^{(2)}$ to depend upon the mortgage rate and othe variables, but for simplicity it is also assumed to be constant over time. To summarize, our model of prepayment behavior calls for the k-th borrower to prepay at time $\tau := \min\{\tau_1, \tau_2\}$, where $\tau_i := \min\{n : u_n^{(i)} \geq d_k^{(i)}\}(i = 1, 2)$.

Now let us specify the models for the three factors: $\{r_n\}$, $\{R_n\}$ and $\{P_n\}$. We assume that the monthly spot rate process $\{r_n\}$ and the mortgage rate process $\{R_n\}$ follow a discrete time Vasicek model:

$$\Delta r_n = \theta_0^{(0)}(\theta_1^{(0)} - r_{n-1})h + \theta_2^{(0)}\sqrt{h}\varepsilon_n^{(0)}, \tag{3.3}$$

$$\Delta R_n = \theta_0^{(1)}(\theta_1^{(1)} - R_{n-1})h + \theta_2^{(1)}\sqrt{h}\varepsilon_n^{(1)}, \tag{3.4}$$

where $\Delta r_n := r_n - r_{n-1}$, $\Delta R_n := R_n - R_{n-1}$, $h = \frac{1}{12}$, and the θ_i^j's are various scalar parameters. The house price process $\{P_n\}$ follows the model:

$$P_n = P_{n-1} \exp(\mu_{n-1} h + \sigma \sqrt{h} \varepsilon_n^{(2)}), \tag{3.5}$$

$$\mu_{n-1} = \phi \mu_{n-2} + (1 - \phi) \log \left(\frac{P_{n-1}}{P_{n-2}} \right), \tag{3.6}$$

where the volatility σ is assumed to be constant and the parameter ϕ satisfies $0 \leq \phi \leq 1$. The drift process μ, driven by past values of itself as in (3.6), is called an exponentially smoothing model. The value $1 - \phi$ is the proportion of a recent change in price brought into a change in the drift. The greater $1 - \phi$ is, the more volatile the drift is, though it depends on the volatility σ. Here innovations $\boldsymbol{\varepsilon} = (\varepsilon_n^{(0)}, \varepsilon_n^{(1)}, \varepsilon_n^{(2)})$ are assumed to be iid (independently and identically distributed) as 3-dimensional normal random variables with mean $\mathbf{0}$ and covariance matrix $\mathbf{\Lambda}$, where

$$\mathbf{\Lambda} := \begin{pmatrix} 1 & \rho_{01} & \rho_{02} \\ \rho_{10} & 1 & \rho_{12} \\ \rho_{20} & \rho_{21} & 1 \end{pmatrix}.$$

Next we specify our model of the distribution of the thresholds $\{(d_k^{(1)}, d_k^{(2)}) : k = 1, \ldots, K\}$. This distribution plays an important role in the valuation of an MBS because the prepayment behavior and thus the cash flow pattern depend upon it and the realizations of $\{(R_n, P_n) : n = 1, \ldots, N\}$. We make the following

ASSUMPTION. The K pairs of random variables $\{(d_k^{(1)}, d_k^{(2)}) : k = 1, \ldots, K\}$ are independentically and identically distributed with common 2-dimensional normal distribution $N(\boldsymbol{\mu}, \mathbf{\Sigma})$, where

$$\boldsymbol{\mu} := \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \end{pmatrix} \quad \text{and} \quad \mathbf{\Sigma} := \begin{pmatrix} (\sigma^{(1)})^2 & \sigma^{(1)}\sigma^{(2)}\delta \\ \sigma^{(1)}\sigma^{(2)}\delta & (\sigma^{(2)})^2 \end{pmatrix}. \tag{3.7}$$

There are at least two ways to think about this assumption. First, you can imagine a large population of potential mortgagors, with the thresholds in this population having the indicated normal distribution, and with the mortgage holders in the MBS pool being a random sample from this population. Alternatively, you can imagine having some detailed knowledge about the specific individuals in the MBS pool. From this knowledge you derive individual threshold distributions for each mortgagor, or even determine specific values of individual thresholds. And then the thresholds for the mortgagors in the pool are approximated by the bivariate normal distribution in our assumption. Either way, a mortgagor randomly selected from the MBS pool is assumed to have threshold values that are consistent with the indicated bivariate normal distribution.

This completes the specification of our model. In principle, from the various specified elements it is possible to derive the probability distributions of the prepayment proportions L_n/K, and using these one can calculate the theoretical value of the MBS by computing the expected values in (2.9). However, in practice this procedure is too complicated, so we find it necessary to estimate the expectations in (2.9) using Monte Carlo simulation, as will be explained in the following section.

4 Monte Carlo Simulation: Method

To value an MBS at initial time $m = 0$ using Monte Carlo simulation and expression (2.9), first observe that the random variable L_n can be expressed as

$$L_n = \sum_{k=1}^{K} L_{kn},$$

where the new random variable $L_{kn} := 1$ if the kth borrower prepays by time n, whereas $L_{kn} := 0$, otherwise. In view of (2.9), therefore, we want to estimate quantities like $E_0^*[\Delta(0,n)L_{nk}]$. Next, observe that, in view of (3.1) and (3.2), the random variable L_{kn} depends upon the random variables $v_n^{(1)}, v_n^{(2)}, d_k^{(1)}$, and $d_k^{(2)}$, where

$$v_n^{(i)} := \max_{j \le n} u_j^{(i)}, \quad i = 1, 2.$$

Since by some fundamental properties of conditional expectations we have

$$E_0^*[\Delta(0,n)L_{kn}] = E_0^*\left[E_0^*[\Delta(0,n)L_{kn}|v_n^{(1)}, v_n^{(2)}]\right] = E_0^*\left[\Delta(0,n)E_0^*[L_{kn}|v_n^{(1)}, v_n^{(2)}]\right],$$

it is apparent that the simulation will be simplified by using an explicit expression for the conditional expectation $E_0^*[L_{kn}|v_n^{(1)}, v_n^{(2)}]$ (this will obviate the need to generate the random variables $d_k^{(1)}$ and $d_k^{(2)}$ in the simulation). The derivation of this expression is straight forward, because we have

$$E_0^*[L_{kn}|v_n^{(1)}, v_n^{(2)}] = P^*(d_k^{(1)} \le v_n^{(1)} \text{ or } d_k^{(2)} \le v_n^{(2)}|v_n^{(1)}, v_n^{(2)})$$

$$= P^*(d_k^{(1)} \le v_n^{(1)}|v_n^{(1)}) + P^*(d_k^{(2)} \le v_n^{(2)}|v_n^{(2)}) - P^*(d_k^{(1)} \le v_n^{(1)}, d_k^{(2)} \le v_n^{(2)}|v_n^{(1)}, v_n^{(2)})$$

$$= \Phi\left(\frac{v_n^{(1)} - \mu^{(1)}}{\sigma^{(1)}}\right) + \Phi\left(\frac{v_n^{(2)} - \mu^{(2)}}{\sigma^{(2)}}\right) - \Phi\left(\frac{v_n^{(1)} - \mu^{(1)}}{\sigma^{(1)}}, \frac{v_n^{(2)} - \mu^{(2)}}{\sigma^{(2)}}, \delta\right),$$

where $\Phi(\cdot)$ denotes the standard normal distribution function and $\Phi(\cdot, \cdot, \delta)$ denotes the bivariate normal distribution for two standard normal random variables having correlation δ.

Now we let $H(\cdot,\cdot)$ denote the real-valued function of two real numbers x and y defined by

$$H(x,y) := \Phi\left(\frac{x-\mu^{(1)}}{\sigma^{(1)}}\right) + \Phi\left(\frac{y-\mu^{(2)}}{\sigma^{(2)}}\right) - \Phi\left(\frac{x-\mu^{(1)}}{\sigma^{(1)}}, \frac{y-\mu^{(2)}}{\sigma^{(2)}}, \delta\right),$$

and note this function is independent of both the time period n and the mortgagor's index k. It follows from (2.9) and the preceding equations that the initial value of the MBS is given by

$$V(0,N) = \sum_{n=1}^{N} E_0^*\left[\Delta(0,n)\left(a_n\left[1-\frac{KH(v_n^{(1)},v_n^{(2)})}{K}\right]+b_n\left[1-\frac{KH(v_{n-1}^{(1)},v_{n-1}^{(2)})}{K}\right]\right)\right]$$

$$= E_0^*\left[\sum_{n=1}^{N}\Delta(0,n)\left(a_n\left[1-H(v_n^{(1)},v_n^{(2)})\right]+b_n\left[1-H(v_{n-1}^{(1)},v_{n-1}^{(2)})\right]\right)\right].$$

It should be noted that $V(0,N)$ is independent of the pool size K.

We will now briefly describe how we use Monte Carlo simulation to estimate $V(0,N)$. First we generate N 3-dimensional vectors of random numbers I times:

$$(\boldsymbol{\varepsilon}_1^{(i)}, \boldsymbol{\varepsilon}_2^{(i)}, \ldots, \boldsymbol{\varepsilon}_N^{(i)}) \quad (i=1,\ldots,I), \tag{4.1}$$

where

$$\boldsymbol{\varepsilon}_j^{(i)} = \begin{pmatrix} \varepsilon_j^{(0)(i)} \\ \varepsilon_j^{(1)(i)} \\ \varepsilon_j^{(2)(i)} \end{pmatrix} \sim iid\ N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho_{01} & \rho_{02} \\ \rho_{10} & 1 & \rho_{12} \\ \rho_{20} & \rho_{21} & 1 \end{pmatrix}\right) \quad (j=1,\ldots,N). \tag{4.2}$$

Through (3.3), (3.4) and (3.5), these innovations in turn generate I paths of spot rates, mortgage rates and house prices, respectively, over the N months. For instance, for the ith set of paths we have

$$(r_1^{(i)}, r_2^{(i)}, \ldots, r_N^{(i)}), (R_1^{(i)}, R_2^{(i)}, \ldots, R_N^{(i)}), (P_1^{(i)}, P_2^{(i)}, \ldots, P_N^{(i)}). \tag{4.3}$$

From these paths we obtain I sets of N random discount factors and I paths of maxima $v_n^{(1)(i)}$ and $v_n^{(2)(i)}$ via (3.1) and (3.2):

$$(\Delta_1^{(i)}, \Delta_2^{(i)}, \ldots, \Delta_N^{(i)}) \quad \text{with} \quad \Delta_n^{(i)} = \exp\left(-\sum_{j=0}^{n-1} r_j^{(i)} h\right), \quad \text{and} \tag{4.4}$$

$$(v_1^{(j)(i)}, v_2^{(j)(i)}, \ldots, v_N^{(j)(i)}) \quad (j=1,2). \tag{4.5}$$

Substituting each of these I sets of random variables in the expression

$$\sum_{n=1}^{N} \Delta(0,n)\left(a_n\left[1-H(v_n^{(1)},v_n^{(2)})\right]+b_n\left[1-H(v_{n-1}^{(1)},v_{n-1}^{(2)})\right]\right)$$

provides I realizations of the "discounted cash flow." Finally we take the simple arithmetic average of these I realizations as our estimate of the expected value $V(0,N)$. In our simulation, we set $I = 1000$ for each case.

5 Monte Carlo Simulation: Numerical Results

In this section we report on a variety of simulation results. These simulation experiments not only demonstrate the potential of our MBS model, and thus ones like it, for practical use, but they provide via comparisons some economic insight about how individual model inputs, such as interest rate volatility and incentive threshold means, affect MBS prices.

Our first simulation experiment is our standard, baseline case. The input parameters are not based upon actual data. Instead, they are chosen to be representative of real-world values. Later in this section we conduct a comparative statics analysis by varying individual parameters.

For our baseline case we consider a 30-year MBS with a \$100 face value and a 6.5% coupon made of mortgage loans with a 7% mortgage rate and equal monthly payments. The 0.5% difference is the servicing fee, and so we have

$$R_0 = 0.07,\ S = 0.005,\ C = 0.065,\ \text{and}\ N = 360.$$

For the parameters in the spot rate model, mortgage rate model, and house price model, for our standard case we chose

$$\begin{aligned}
&\theta_0^{(1)} = 0.2,\ \theta_1^{(1)} = 0.05,\ \theta_2^{(1)} = 0.008,\ r_0 = 0.05,\\
&\theta_0^{(2)} = 0.2,\ \theta_1^{(2)} = 0.07,\ \theta_2^{(2)} = 0.016,\\
&P_0 = 100,\ \mu_0 = 0.00,\ \sigma_0 = 0.06,\ \phi = 0.5,\\
&\rho_{01} = 0.8,\ \rho_{02} = 0.5,\ \rho_{12} = 0.7,\ \text{and}\ h = 1/12 = 0.083.
\end{aligned}$$

In this standard case, the short-term rate $\{r_n\}$ and mortgage rate $\{R_n\}$ are distinguished in two ways: the volatility $\theta_2^{(1)} = 0.016$ (i.e. 1.6% annual rate) of $\{R_n\}$ is twice the short rate volatility $\theta_2^{(0)} = 0.008$, and the mean reversion levels are respectively $\theta_1^{(1)} = 0.07$ and $\theta_1^{(0)} = 0.05$ (so mortgage rates will usually be higher than short rates). The adjustment speed to the mean reversion level is commonly 0.20. The two rates are highly correlated (0.8). On the other hand, the volatility for house price changes is 6% and the initial drift is set to 0. The correlations of $\{\log P_n\}$ to $\{R_n\}$ and $\{r_n\}$

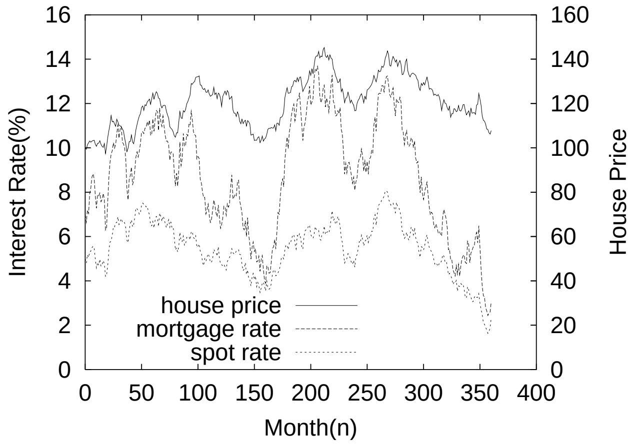

Figure 1: Sample paths of spot rate, mortgage rate and house price process

are respectively 0.7 and 0.5 in their innovations. As a demonstration, a realization of the sample paths of the spot rate, mortgage rate and house price processes is graphed in Figure 1.

Next, the parameters of the normal distribution of the thresholds are given as

$$\mu^{(1)} = 0.04,\ \sigma^{(1)} = 0.0133,$$
$$\mu^{(2)} = 0.4,\ \sigma^{(2)} = 0.133, \text{ and } \delta = 0.5.$$

With this specification and an initial mortgage rate of 7%, the median mortgagor will refinance when the mortgage rate hits 3%. Similarly, with an initial house price of $100, the median mortgage holder will sell his property when its price increases approximately 40%.

For this standard, baseline case we obtained a theoretical value of the MBS as 111.948 dollars. The values of cash flows $\{CF(0,n)\}$ with and without prepayments in this Monte Carlo evaluation are graphed in Figure 2. The graph shows that the values of the cash flows in the first 50 months overwhelmingly dominate those in later months. This implies significant prepayments in the early months. The average percentage of borrowers who prepaid due to either sale of house or refinancing is approximately 66%.

5.1 Comparison of one-factor, two-factor and three-factor models

In this subsection we compare the one-factor model in KK(2000) which assumed the linear relation $R_n = a + br_n$, a two-factor model for refinancing in which the roles of the short-term rate $\{r_n\}$ and the mortgage rate $\{R_n\}$

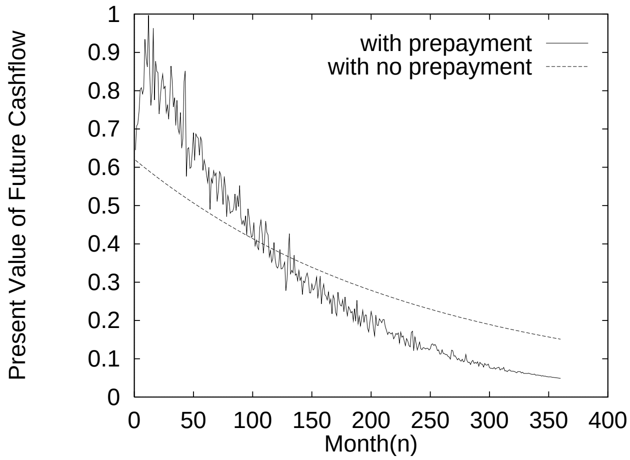

Figure 2: The values of cash flows without and with prepayments

Table 1: price vs. model

model	price(1)	price(2)	price(3)	price(4)	price(5)
one-factor	111.836	111.761	111.557	111.962	111.912
two-factor	112.379	112.217	111.957	112.343	112.255
three-factor	111.948	111.752	111.506	111.877	111.779

are separated, but there is no incentive factor associated with rising property prices, and the preceding baseline case for our three-factor model.

We specified the threshold unit as 0.004 in the case of one-factor model because the volatility of the short term rate is a half of that of the mortgage rate. We ran 5 independent simulations each of which consists of 1000 paths of $\{r_n\}$, $\{R_n\}$ and $\{P_n\}$. The result is summarized in Table 1.

From the table it is observed that the separation of the roles of the discount factor by $\{r_n\}$ and the refinancing incentive factor by $\{R_n\}$ increases MBS prices, while the addition of the equity incentive to the two-factor model decreases MBS prices. The values of the MBS in the one-factor and three factor models are close each other in these simulations.

5.2 Effect of (μ_0, σ, ϕ) on MBS prices

Here we investigate some effects of changes of the house price parameters (μ_0, σ, ϕ) on MBS values. We present three cases, where for each we fix one parameter at its baseline value and vary the other two.

(1) Effect of changes in (μ_0, σ) with $\phi = 0.5$ (Table 2)

(2) Effect of changes in (σ, ϕ) with $\mu_0 = 0.0$ (Table 3)

(3) Effect of changes in (ϕ, μ_0) with $\sigma = 0.06$ (Table 4).

Both Table 2 and 3 show that the MBS prices decrease as σ increases with the other two parameters held fixed. This is natural because an increase in volatility increases the possibility of more prepayments. On the other hand, these two tables also show that the prices are insensitive to changes of the other parameters (μ_0, ϕ) when the volatility is held fixed.

From Table 4 where $\sigma = 0.06$, it is observed that when ϕ is fixed, MBS prices go down as μ_0 increases. This is because an increase in μ_0 tends to increase the appreciation or drift rate for housing prices, thereby leading to more prepayments. This tendency is stronger when ϕ is larger as in Table 4 and Figure 3. Recall that larger ϕ's introduce less information on new changes of the house price into the drift movement, making the drift movement more stable. This is confirmed in Figure 4 where the graphs of the drift movements are presented for the two cases $\phi = 0.1$ and $\phi = 0.9$, both with $\mu_0 = 0.09$. Thus when ϕ is larger, the initial drift μ_0 has a greater effect on the movement of house prices and hence on MBS prices, as is shown in Figure 5.

5.3 Effect of the correlations $(\rho_{01}, \rho_{02}, \rho_{12})$ on MBS prices

The correlations $(\rho_{01}, \rho_{02}, \rho_{12})$ of the innovations of $\{r_n\}$, $\{R_n\}$ and $\{P_n\}$ also affect MBS prices. To investigate this effect, simulations with proportional changes of the correlations and with the other parameters fixed were carried out in Table 5. The table shows that as the correlations increase, the prices decrease rather greatly, as one might expect. When ρ_{12} is positive, the incentive variables $u_n^{(1)} = R_0 - R_n$ and $u_n^{(2)} = \log P_n - \log P_0$ are negatively correlated, and when ρ_{12} is higher, the correlation of ($u_n^{(1)}$ and $u_n^{(2)}$) tends closer to -1 leading to larger $v_n^{(1)} = \max_{j \le n}\{u_j^{(1)}\}$ and $v_n^{(2)} = \max_{j \le n}\{u_j^{(2)}\}$. This situation is demonstrated in Figure 6

5.4 Effect of threshold parameters on MBS prices

The threshold correlation δ also affects the prepayment ratios in the model and hence the MBS prices. As δ increases, the thresholds $(d_k^{(1)}, d_k^{(2)})$'s of borrowers get closer to the 45 degree line (see Figure 7). But then paths of $(u_n^{(1)}, u_n^{(2)})$ have to move more widely to cause a certain level of prepayment. Hence when δ is higher, $(u_n^{(1)}, u_n^{(2)})$ tends to cause less prepayments, meaning higher MBS prices. This situation is confirmed in Table 6.

We also change the mean level $(\mu^{(1)}, \mu^{(2)})$ of thresholds and see the effect on MBS prices. Clearly the higher the mean level is, the less prepayments occur, and hence the larger the MBS prices are. This is demonstrated in

Table 2: price vs. μ_0 and σ ($\phi = 0.5$)

$\mu_0 \setminus \sigma$	0.02	0.04	0.06	0.08	0.1
-0.06	112.369	112.266	111.981	111.534	110.995
-0.03	112.369	112.258	111.967	111.507	110.963
0.00	112.367	112.249	111.948	111.470	110.919
0.03	112.364	112.237	111.924	111.436	110.872
0.06	112.362	112.227	111.902	111.400	110.829
0.09	112.359	112.214	111.880	111.365	110.785

Table 3: price vs. σ and ϕ ($\mu_0 = 0.0$)

$\sigma \setminus \phi$	0.1	0.2	0.3	0.4
0.02	112.367	112.367	112.367	112.367
0.04	112.248	112.249	112.249	112.249
0.06	111.944	111.945	111.945	111.946
0.08	111.465	111.466	111.467	111.468
0.1	110.905	110.907	110.912	110.913

0.5	0.6	0.7	0.8	0.9
112.367	112.367	112.367	112.367	112.367
112.249	112.249	112.250	112.251	112.252
111.948	111.950	111.951	111.954	111.961
111.470	111.472	111.475	111.482	111.506
110.919	110.926	110.935	110.941	110.971

Table 4: price vs. ϕ and μ_0 ($\sigma = 0.06$)

$\phi \setminus \mu_0$	-0.06	-0.03	0	0.03	0.06	0.09
0.1	111.966	111.958	111.944	111.932	111.918	111.905
0.2	111.969	111.959	111.945	111.930	111.913	111.902
0.3	111.972	111.961	111.945	111.930	111.909	111.896
0.4	111.975	111.964	111.946	111.928	111.906	111.891
0.5	111.981	111.967	111.948	111.924	111.902	111.880
0.6	111.992	111.971	111.950	111.921	111.892	111.856
0.7	112.010	111.978	111.951	111.911	111.872	111.830
0.8	112.036	111.998	111.954	111.899	111.833	111.758
0.9	112.114	112.041	111.961	111.840	111.700	111.546

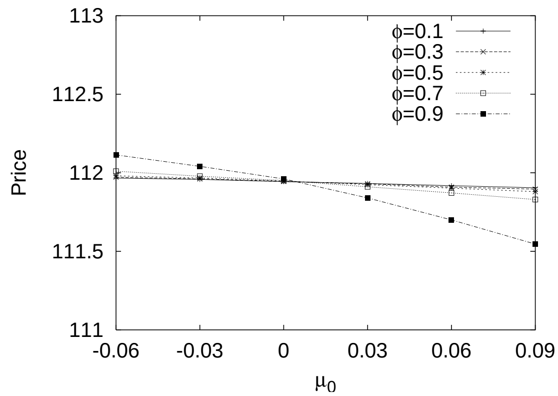

Figure 3: price vs. μ_0

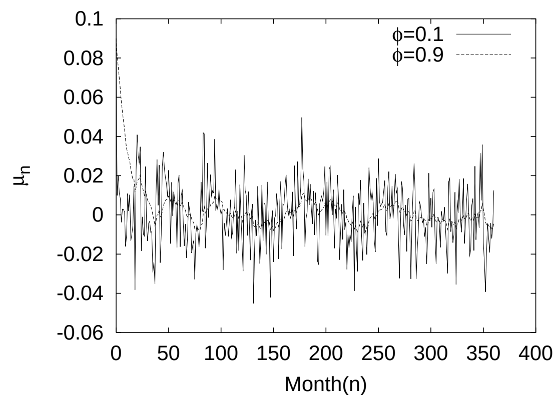

Figure 4: sample paths of $\{\mu_n\}(\mu_0 = 0.09)$

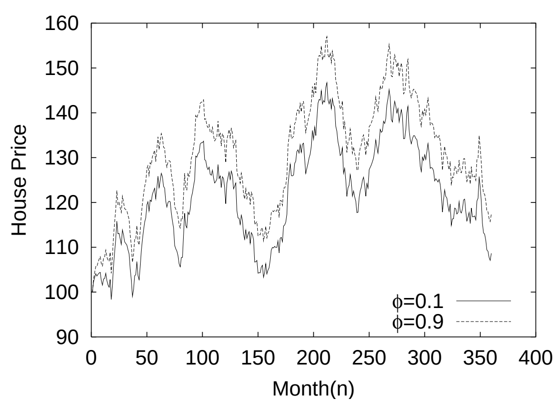

Figure 5: sample paths of house price process($\mu_0 = 0.09$)

Table 5: price vs. correlations between spot rate, mortgage rate, and house price

ρ_{01}	ρ_{02}	ρ_{12}	price
0.0	0.0	0.0	113.425
0.2	0.125	0.175	113.049
0.4	0.25	0.35	112.658
0.6	0.375	0.525	112.290
0.8	0.5	0.7	111.948

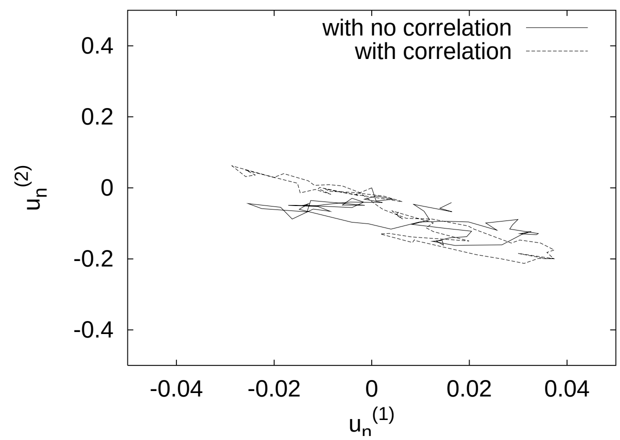

Figure 6: sample paths of $(u_n^{(1)}, u_n^{(2)})$ ($\rho_{12} = 0$ and $\rho_{12} = 0.7$)

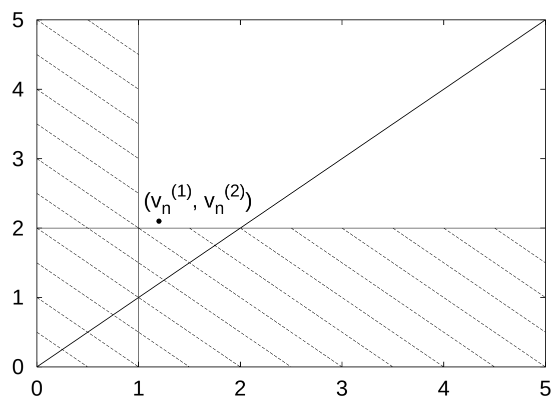

Figure 7: Prepaid area

Table 6: price vs. correlation between thresholds (δ)($\mu_0 = 0,\ \sigma_0 = 0.06,\ \phi = 0.5$)

δ	price	δ	price
0.0	111.860	0.5	111.948
0.1	111.877	0.6	111.967
0.2	111.894	0.7	111.986
0.3	111.911	0.8	112.005
0.4	111.929	0.9	112.025
		1.0	112.052

Table 7: price vs. $\mu^{(1)}$ and $\mu^{(2)}$

$\mu^{(1)} \setminus \mu^{(2)}$	0.2	0.3	0.4	0.5	0.6
0.02	104.857	105.357	105.558	105.639	105.680
0.03	107.817	108.665	109.015	109.159	109.232
0.04	110.331	111.467	111.945	112.143	112.246
0.05	112.112	113.453	114.017	114.254	114.378
0.06	113.242	114.722	115.343	115.607	115.746

Table 7 and Figure 8. The prices are shown to be very sensitive to changes of the mean level.

5.5 Effect of interest model parameters on MBS prices

Here we change the volatility parameters $(\theta_2^{(0)}, \theta_2^{(1)})$ of the interest rates with the correlation $\rho_{01} = 0.8$ held fixed. Since the maximum $v_n^{(1)} = \max_{j \le n} u_j^{(1)}$ matters for pricing MBS's, an increase in the volatility $\theta_2^{(1)}$ tends to increase $v_n^{(1)}$ and hence decreases MBS values. On the other hand, since $\{r_n\}$ acts as a discount factor for cash flows, the effect of $\theta_2^{(0)}$ on MBS prices is indefinite. This is shown in Table 8. In the case of $\theta_2^{(1)} = 0.008$ in the table, as $\theta_2^{(0)}$ increases, the MBS prices gradually increase, while in case of $\theta_2^{(1)} = 0.012$, the MBS prices first decrease and then increase.

Next we consider the effect of the adjustment speed parameters $(\theta_0^{(0)}, \theta_0^{(1)})$ on MBS prices when the mean reversion levels $(\theta_1^{(0)}, \theta_1^{(1)}) = (0.05, 0.07)$ and the volatilities $(\theta_2^{(0)}, \theta_2^{(1)}) = (0.008, 0.016)$ are fixed. Table 9 gives a result for this case. When the adjustment speed to the mean reversion level is bigger, the interest rates tend to stay around the mean reversion level, implying less prepayments, though the volatility is another factor. This phenomenon is demonstrated in Figure 9. As seen in Table 9, with $\theta_0^{(0)}$ fixed, the MBS

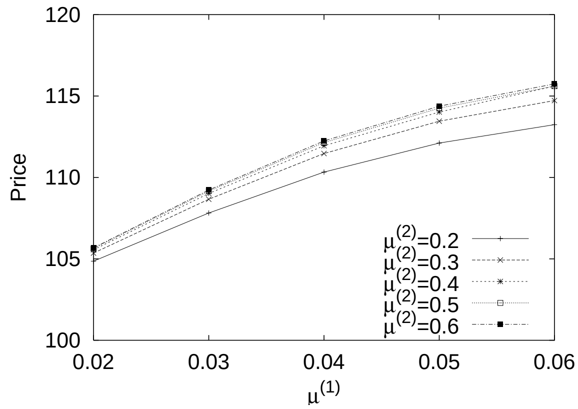

Figure 8: price vs. $\mu^{(1)}$ and $\mu^{(2)}$

Table 8: price vs. $\theta_2^{(0)}$ and $\theta_2^{(1)}$

$\theta_2^{(0)} \setminus \theta_2^{(1)}$	0.008	0.012	0.016	0.02
0.004	116.141	114.454	112.318	110.316
0.006	116.323	114.443	112.114	109.973
0.008	116.566	114.480	111.948	109.660
0.01	116.871	114.566	111.819	109.376

Table 9: price vs. $\theta_0^{(0)}$ and $\theta_0^{(1)}$

$\theta_0^{(0)} \setminus \theta_0^{(1)}$	0.1	0.2	0.3	0.4
0.1	110.516	111.684	112.717	113.678
0.2	111.016	111.948	112.773	113.540
0.3	111.326	112.154	112.885	113.564
0.4	111.526	112.298	112.978	113.608

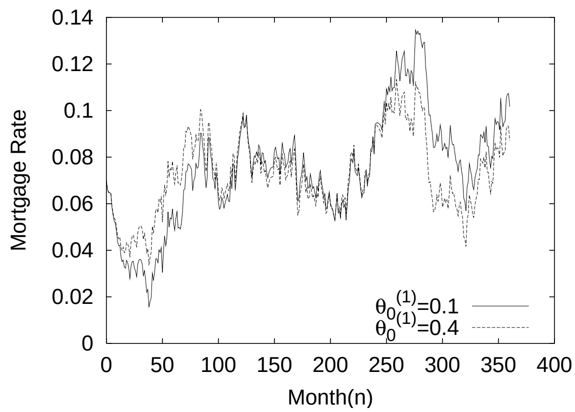

Figure 9: sample paths of mortgage rate processes

prices increase as the speed $\theta_0^{(1)}$ for $\{R_n\}$ increases, while for $\theta_0^{(1)}$ fixed the effect of the speed $\theta_0^{(0)}$ for discount factor $\{r_n\}$ on MBS prices is indefinite and does not change the prices much.

6 Concluding Remarks

In this paper we extended KK(2000)'s framework and model in a discrete time setting to describe prepayment behavior due to the incentives of the equity factor for rising house prices and interest rate factor for refinancing. The extension involves a non-Markovian model of house prices and leads to a two-dimensional boundary hitting problem for two factors. In addition, we separated the role of the short term interest rate as the discount factor from that of the mortgage interest rate as an incentive factor associated with prepayment. Furthermore, the prepayment behavior is directly embedded into the cash flows of the MBS; this is important because cash flow patterns and hence values of MBS are changed by the distribution of the prepayment times.

Through various simulations we found that our methodology has great potential to produce a practical and useful valuation model for mortgage backed securities. But our specific model is only representative of the general framework that we are developing. In the course of specifying and testing our specific model we ignored important issues such as partial prepayment, default, and sale of house caused by exogenous, non-economic reasons. Further, we assumed that thresholds for incentive variables are constant, whereas to be more realistic they should depend upon time and possibly other variables. Most importantly, the input parameters in our simulation experiments were hypothetical rather than based upon actual data. It would be desirable to use actual data to calibrate and validate our model, thereby possibly revealing additional and desirable extensions and modifications. All these problems and issues call for further research.

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