Optimal Mortgage Refinancing with Endogenous Mortgage Rates: an Intensity Based, Equilibrium Approach[*]

Stanley R. Pliska
Department of Finance, University of Illinois at Chicago
601 S. Morgan Street, Chicago, IL 60607-7124 USA
e-mail: srpliska (at) uic.edu

Abstract

This paper summarizes recent research on a new approach, namely, an equilibrium approach, to the valuation of fixed-rate mortgage contracts. Working in a discrete time setting with the mortgagor's prepayment behavior described by a suitable intensity process and with exogenous mortgage rates, the value of the contract is derived in an explicit form that can be interpreted as the principal balance plus the value of a certain swap. This leads to a nonlinear equation for what the mortgage rate must be in a competitive market, and thus mortgage rates are endogenous and depend upon the mortgagor's prepayment behavior. The complementary problem, where mortgage rates are exogenous and the mortgagor seeks the optimal refinancing strategy, is then solved via a Markov decision chain. Finally, the equilibrium problem, where the mortgagor is a representative agent in the economy who seeks the optimal refinancing strategy and where the mortgage rates are endogenous, is developed, solved, and analysed. Existence and uniqueness results, as well as a numerical example, are provided.

Keywords: mortgage valuation, endogenous mortgage rates, equilibrium, Markov decision chain, dynamic programming, intensity process, hazard rate

Contents

[*]I am grateful to NM Rothschild & Sons for its generous support in the form of a Rothschild Visiting Professorship at the Isaac Newton Institute's 2005 Programme on Developments in Quantitative Finance. I thank Tomasz Bielecki, Charles Saunders, and Warren Nichols for useful comments.

1 Introduction

While there is widespread agreement that the value of a mortgage contract subject to prepayment but not default risk should be given by an expectation of the present value of the cash flow, the devil is in the details. A wide variety of approaches have been considered, most of which are commonly classified into one of two categories. One kind of approach has been variously called a reduced form approach, an exogenous approach, an empirical approach, and an econometric approach. The basic idea is to build a stochastic model for interest rates and possibly other economic factors, and then add a statistical model describing how the mortgagor's prepayment behavior depends on the factors. While such an overall model can be quite complicated, it is usually straightforward to use Monte Carlo simulation to estimate the expected value of the discounted cash flow. Some of the many papers in this category are by Schwartz and Torous [17], [18], Deng [1], Deng, Quigley, and Van Order [2], Gorovoi and Linetsky [7], Kariya and Kobayashi [9], Kariya, Pliska, and Ushiyama [10], and Kau, Keenan, and Smurov [11].

Of significance is that in some of this research, dating back at least to Schwartz and Torous [17], [18], it was recognized that the random time when a mortgagor prepays can be described with a hazard rate model, that is, the conditional rate of prepayment given the current state of any factors and no prepayment to date. Perhaps this development was inspired by the engineering literature on reliability theory, as the time of mortgage prepayment is clearly analogous to the failure time of a system. In any event, Schwartz and Torous [17] used this hazard rate viewpoint in conjunction with a two-factor model in order to present a partial differential equation for the value of a mortgage contract. Moreover, as will be seen in this paper, recent developments involving the hazard rate as a model of a default time in the credit risk literature lead to new results, involving intensity processes, for mortgage contract valuation.

The other main kind of approach for the valuation of mortgage contracts is called an option-based or a structural approach. The basic idea is to incorporate some kind of optimal behavior with respect to the mortgagor's decision about when to refinance. Moreover, the way to do this is to appeal to some intuition based upon the theory of the optimal early exercise decision for American options, usually leading to a recursive valuation procedure that resembles the one used for the binomial option pricing model. For example, Kalotay, Yang, and Fabozzi [8] described the following procedure: first build an interest rate lattice, and then, starting with the final scheduled cash flows of the mortgage, work backwards through the lattice computing the mortgage's value, comparing the value with no refinancing and the value of a newly refinanced mortgage, with the latter assumed to be par plus the refinancing cost. If the latter is less, then the value of the existing mortgage at the node is replaced by the value of the new mortgage. Some others who took an option-based approach are Dunn and McConnell [3], [4], Stanton [19], Nakagawa and Shouda [13], Stanton and Wallace [20], Dunn and Spatt [5], and Longstaff [12].

The latter three papers are noteworthy because, in contrast to all the other option-based papers which assumed mortgage rates are exogenous, Stanton and Wallace [20], Dunn and Spatt [5], and Longstaff [12] allowed for endogenous values of fixed rate mortgages. These authors studied discrete time, finite horizon models, with time equal to the age of the mortgage contract. The mortgage rates were computed recursively, much like the "binomial option pricing" procedure by Kalotay, Yang, and Fabozzi [8] that was described above. But the model assumptions made by Stanton and Wallace [20], Dunn and Spatt [5], and Longstaff [12] are unclear, due in part to the limited use of mathematics in their expositions. Suffice it to say that their models are significantly different from the one in this paper, as evidenced by the fact that their endogenous mortgage rates seem to depend upon the age of the mortgage contract.

As indicated above, the theory of hazard rates and intensity processes for modelling default times in the credit risk literature has advanced considerably in recent years. Since the time of a default is analogous to the time when a mortgage balance is prepaid, it was natural to translate some of the credit risk developments to mortgage valuation. This was recently accomplished by Goncharov [6], who worked entirely in a continuous time setting. In particular, he showed how to unify the reduced form and the option-based approaches, he derived some explicit formulas for a mortgage's value, he derived a variety of partial differential equations useful for computing mortgage values, and he used an explicit valuation formula to provide a nonlinear equation for the endogenous mortgage rate.

This paper makes several contributions. First, some intensity based valuation results that Goncharov [6] derived for a continuous time environment are here derived for a discrete time financial market. In particular, with the mortgagor's prepayment behavior described by a suitable intensity process and with exogenous mortgage rates, in Section 2 the value of the contract is derived in an explicit form that can be interpreted as the principal balance plus the value of a certain swap. This leads in Section 3 to a nonlinear equation for what the mortgage rate must be in a competitive market, and thus mortgage rates are endogenous and depend upon the mortgagor's prepayment behavior. The complementary problem, where mortgage rates are exogenous and the mortgagor seeks the optimal refinancing strategy, is then solved in Section 4 via a Markov decision chain. Various theoretical results about computational algorithms and existence of solutions are included. The equilibrium problem, where the mortgagor is a representative agent in the economy who seeks the optimal refinancing strategy and where the mortgage rates are endogenous, is developed, solved, and analyzed in Section 5. In particular, the existence of an equilibrium solution is established. Section 6 provides a simple computational example that illustrates various theoretical points, although it is probably not realistic enough to draw conclusions about actual mortgage markets. Section 7 provides some concluding remarks, summarizing our main economic conclusions and suggesting some good directions for future research. Finally, an appendix contains technical derivations and proofs of many of the main results.

It should be noted that [15] is a preliminary, proceedings version of the present paper.

2 Valuation of Mortgage Contracts

This paper focuses on the valuation of fixed rate mortgage contracts having N contracted coupon payments each of amount c dollars. If m is the mortgage rate at contract initiation (this interest rate is expressed on a per payment period, not necessarily on an annual, basis) and if $P(n, m)$ denotes the principal balance immediately after the nth coupon payment is made, then by simple time value of money considerations

$$P(n+1, m) = (1+m)P(n, m) - c. \tag{1}$$

Using this equation recursively one obtains

$$P(n, m) = \frac{c}{m}[1-(1+m)^{-k}] + \left(\frac{1}{1+m}\right)^k P(n+k, m), \quad k = 1, ..., N-n. \tag{2}$$

Throughout this paper it will be assumed that the mortgage contracts are fully amortizing, that is, $P(N, m) = 0$, and so (2) implies

$$P(n, m) = \frac{c}{m}[1-(1+m)^{n-N}], \quad n = 0, 1, ..., N. \tag{3}$$

In particular, since $P(0, m)$ is the initial principal, the contracted coupon payment c is given in terms of the maturity N of the mortgage and the contracted mortgage rate m by

$$c = \frac{mP(0, m)}{1-(1+m)^{-N}}. \tag{4}$$

In accordance with common practice, just after any coupon payment the mortgagor can pay the principal balance, thereby terminating the mortgage contract. For simplicity it will be assumed that the mortgagor cannot pay any amount greater than the contracted coupon payment c unless it is the entire principal balance. Thus it will be assumed that immediately after the nth coupon payment the mortgagor must either pay the principal balance $P(n, m)$ or continue with the existing mortgage contract at least one more period. Also, since the focus of this paper is on the prepayment option, none of the mortgages considered here are subject to default.

From the perspective of the mortgage lender or of a third party considering the purchase of the mortgage contract, the value of the mortgage contract equals the expectation of the discounted cash flow up through the prepayment time or N, whichever is less. In accordance with standard financial valuation practice, the discounting is respect to the riskless, one-period short rate and the expectation is respect to a risk neutral probability measure. So to model this it will be assumed there is a probability space $(\Omega, \mathcal{F}, Q, \{\mathcal{F}_t\}_{t\geq 0})$, where Q is a *risk neutral probability measure* and where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration describing how interest rate information and possibly other economic information (but not information about whether prepayment has occurred) is revealed to market participants. The *riskless interest rate process* $r = \{r_t; t = 1, 2, ...\}$ is a predictable process, where r_t represents the one-period riskless interest rate for loans from time $t-1$ to time t. The mortgagor's prepayment time, τ, is a *stopping time* that takes one of the values $1, 2, ..., N$, with $\tau = N$ meaning the mortgage is not prepaid early. Thus the value V of the mortgage contract at contract initiation is given by risk neutral valuation to be

$$V = E[\frac{c}{1+r_1} + \frac{c}{(1+r_1)(1+r_2)} + \ldots + \frac{c}{(1+r_1)\ldots(1+r_{\tau\wedge N})} + \frac{P(\tau\wedge N, m)}{(1+r_1)\ldots(1+r_{\tau\wedge N})}]. \quad (5)$$

A similar equation holds for the value of the contract at times subsequent to contract initiation.

Remark 2.1 To develop a practical model like this, one can imagine starting with a spot rate model of riskless interest rates that includes various securities such as the usual bank account process and zero coupon bonds of all maturities. With the mortgage contract and the issue of prepayment excluded, this model may or may not be complete. For example, see the lattice, Markov chain models in Pliska [14]. Then one would add to this model the mortgage contract together with the random prepayment time τ. If the initial riskless interest rate model is complete, then the final model could be too, in which case one has all the usual implications about uniqueness of the risk neutral probability measure and replication of contingent claims. But even if the model is not complete, the model will be free of arbitrage opportunities that can be obtain by trading the various securities.

To develop an expression for the contract value that is more useful than (5) it is convenient to introduce the *prepayment intensity process* $\gamma = \{\gamma_t; t = 1, 2, \ldots, N\}$, where

$$\gamma_t := Q(\tau = t|\tau \geq t, \mathcal{F}_t), \quad t = 1, 2, ..., N.$$

Thus γ_t can be interpreted as the conditional, risk neutral probability that prepayment will occur this period given the current history of economic information and given the fact that prepayment did not occur earlier. Note that since the short rate process r is predictable, it is possible that the event $\{\tau = t\}$ depends on the value of the short rate that will apply between times t and $t+1$.

Using this intensity process, expression (5), and basic properties of conditional expectation (see the appendix for for the details) one obtains the following result:

Proposition 2.1 *The initial value of a mortgage contract is given by*

$$V = E\Big[\sum_{i=1}^{N} \frac{c + \gamma_i P(i, m)}{1+r_1} \prod_{j=2}^{i} \frac{1-\gamma_{j-1}}{1+r_j}\Big], \quad where \prod_{j=2}^{1} := 1. \quad (6)$$

This last expression for the initial value of a mortgage contract is still not very useful. A better formula is obtained by using (1) and (6) together with a lot of algebra (see the appendix), namely:

Theorem 2.1 *The initial value of a mortgage contract is given by*

$$V = \frac{1+m}{1+r_1} P(0,m) + \frac{1}{1+r_1} E\Big[\sum_{i=1}^{N-1} (m - r_{i+1}) P(i,m) \prod_{j=1}^{i} \frac{1-\gamma_j}{1+r_{j+1}}\Big]. \tag{7}$$

Remark 2.2 Note the factor in the first term will be approximately equal to one, so the first term will approximately equal the initial principal. The second term can be interpreted as the discounted value of an amortizing swap, where one party pays the fixed mortgage rate m and the other party pays the floating rate r, and where the swap can terminate in accordance with the random prepayment time τ. Thus the initial value of the mortgage contract is approximately equal to the initial principal plus the value of a swap. A similar expression can be obtained for the value of the mortgage contract at subsequent times: the time-t value of the contract from the perspective of the mortgagee is, up to interest rate factors close to one, equal to the time-t principal balance plus the value of a swap.

3 Endogenous Mortgage Rates

While the mortgage market might be free of arbitrage opportunities that can be achieved by trading fixed income securities and the mortgage contract, it can be vulnerable to another kind of arbitrage opportunity if the mortgage contract value V is less than the initial principal $P(0,m)$. In other words, why would a lending institution offer a loan of $P(0,m)$ in exchange for a cash flow that is worth a strictly smaller amount? On the other hand, the possibility $V > P(0,m)$ implies attractive, profitable lending opportunities for financial institutions, and so it shall be argued that in a competitive market one will have $V = P(0,m)$.

With $V = P(0,m)$ it follows (see the appendix) from (7) that the mortgage rate m is endogenous and must satisfy a nonlinear equation, as summarized in the following:

Theorem 3.1 *In a competitive mortgage market where $V = P(0,m)$ the mortgage rate m is endogenous and satisfies*

$$0 = (m - r_1) P(0,m) + E\Big[\sum_{i=1}^{N-1} (m - r_{i+1}) P(i,m) \prod_{j=1}^{i} \frac{1-\gamma_j}{1+r_{j+1}}\Big] \tag{8}$$

or, equivalently,

$$0 = m - r_1 + E\Big[\sum_{i=1}^{N-1} (m - r_{i+1}) \Big(\frac{(1+m)^N - (1+m)^i}{(1+m)^N - 1}\Big) \prod_{j=1}^{i} \frac{1-\gamma_j}{1+r_{j+1}}\Big]. \tag{9}$$

Remark 3.1 It is important to note that the endogenous value of the mortgage rate m depends upon the mortgagor's prepayment behavior via the intensity process γ. So if a mortgagor changes this prepayment behavior, mortgage market forces should cause the mortgage rates to change. Hence this leads to a certain equilibrium problem, the subject of a later section. Also in a later section attention will focus on a special case of the preceding model where the spot rate process r is a Markov chain, in which case it is appropriate to think of the expectations in (8) and (9) as being conditional on the values of the current state for r_1.

Remark 3.2 One sees that the value of m satisfying equation (8) must be some kind of weighted average of future possible values of the riskless short rate r. In particular, it must be smaller than the biggest possible value of the short rate. This suggests that when the short rate is historically high and when the yield curve is inverted, the endogenous mortgage rate will actually be smaller than the current short rate.

4 Optimal Refinancing

In this section attention is given to a complementary problem where the mortgage rate is an exogenous stochastic process and the mortgagor seeks to refinance the loan in an optimal fashion. In contrast to other approaches in the literature, the mortgagor here might choose to refinance several times before the loan is ultimately paid off.

For tractability it will be assumed that the riskless spot rate process $r = \{r_t; t = 1, 2, \ldots\}$ and the mortgage rate process $M = \{M_t; t = 0, 1, \ldots\}$ together comprise a time-homogeneous Markov chain having a finite state space. Here M_t represents the fixed rate for N-period mortgages that are initiated at time t. This two-component Markov chain can easily be generalized by adding additional factors such as a measure of property value, but this will not be done here for the sake of the exposition.

It is assumed the transaction cost $K(P)$ is incurred if and when the principal balance P is refinanced. Here $K(\cdot)$ is a specified, deterministic function. The decision to prepay is made immediately after the contracted coupon payment c has been paid. If the mortgagor decides to refinance at time t with a current principal balance of P, then at that time an additional $K(P)$ dollars are paid and a new N-period mortgage is initiated at the prevailing mortgage rate M_t. Alternatively, if the mortgagor decides to continue with the current mortgage contract then the prevailing coupon rate of c dollars will be paid for at least one more period. The mortgagor can refinance as many times as desired, but each time the amount of the new loan must equal the principal balance of the contract being terminated. Thus the mortgagor cannot "pull equity out of the property" by refinancing with a bigger loan. Nor can the mortgagor make any periodic payment, either bigger or smaller, than the currently contracted coupon payment c.

It is important to be mathematically precise about the mortgagor's objective. Here it will be assumed that the mortgagor seeks to minimize the expected present value of the cash flow (which includes any transaction costs associated with refinancings) until the loan is paid off. In particular, it is assumed for simplicity that the mortgagor does not prepay early for "external" reasons such as selling the property (although this is a strong assumption that would be nice to abandon). As usual, the expectation here will be with respect to a risk neutral probability measure and discounting will be with respect to the spot rate process r.

In summary, given the Markov chain (r, M) the mortgagor's problem is to find the refinancing schedule having the minimum expected discounted value. Hence it is natural to formulate this problem as a dynamic programming problem, specifically, as a Markov decision chain (see Puterman [16] for a comprehensive treatment of Markov decision chains). For the state variables one then needs to have the current values of the two interest rates as well as some variables describing the status of the current mortgage contract. There are four of the latter variables, namely, the contracted mortgage rate m, the number n of payments already made, the current principal balance P, and the current coupon payment c. However, in view of relationship (3), just three of these four variables need to be kept track of, say just m, n, and P. Hence the Markov decision chain must have five state variables: r, M, m, n, and P.

Now it is straightforward to proceed with the formulation of this Markov decision chain, but with five state variables the result will be in jeopardy of having too many variables for computational tractability. So to reduce the number of state variables, use will be made of Theorem 3.1 which suggests it is appropriate to assume $M_t = m(r_{t+1})$ for some deterministic function $m(\cdot)$ (note that since r is a predictable process, at time t the corresponding value of the short rate process is r_{t+1}). But for $M_t = m(r_{t+1})$ to hold for all t and a single function $m(\cdot)$, it is necessary to assume the time index in equations (8) and (9) refers to the age of the current mortgage contract and not the time on the clock. Thus while the prepayment time can depend upon the age of the current contract and the values of the short rate during the life of the contract, it does not depend upon, for instance, values of the short rate that might have occurred before the initiation of the current contract. Thus for some function $m(\cdot)$ computed from (9) we have $M_t = m(r_{t+1})$ for all t. Moreover, at the cost of incorporating this function in the model, the number of state variables have been reduced to four: r, m, n, and P.

Let $v(n, P, m, r)$ denote the minimum expected discounted value of the cash flow given n payments have been made on a contract having contracted mortgage rate m and remaining principal balance P and given that the riskless rate of interest for the next period is r. If it is optimal for the mortgagor to continue at least one more period with the current contract, then by (1)-(4)

$$v(n, P(n,m), m, r) = (1+r)^{-1}[c + E_r v(n+1, P(n+1,m), m, R)]$$

$$= (1+r)^{-1}\Big[\frac{mP(n,m)}{1-(1+m)^{n-N}} + E_r v(n+1, (1+m)P(n,m)\frac{(1+m)^{N-n-1}-1}{(1+m)^{N-n}-1}, m, R)\Big],$$

where R here is the random short rate next period and the expectation is conditioned on the current value r of the short rate. On the other hand, if it is optimal to refinance, then $v(n, P(n,m), m, r) = K(P(n,m)) + v(0, P(n,m), m(r), r)$. Hence the dynamic programming equation is

$$v(n, P, m, r) = \min\Big\{K(P) + v(0, P, m(r), r),$$

$$(1+r)^{-1}\Big[\frac{m(1+m)^{N-n}}{(1+m)^{N-n}-1}P + E_r v(n+1, \frac{(1+m)[(1+m)^{N-n-1}-1]}{(1+m)^{N-n}-1}P, m, R)\Big]\Big\}. \quad (10)$$

Now this dynamic programming equation can perhaps be solved, but doing so is computationally challenging since there are four state variables, one of which is continuous. So to deal with this it is convenient to make one more assumption: the transaction cost $K(P) = kP$ for some positive constant k. As explained in the appendix, this assumption leads to the following result:

Proposition 4.1 *Suppose $K(P) = kP$. If the function $f(n, m, r)$ satisfies the dynamic programming equation*

$$f(n, m, r) = min\Big\{k + f(0, m(r), r),$$

$$\frac{1+m}{(1+r)[(1+m)^{N-n}-1]}\Big[m(1+m)^{N-n-1} + [(1+m)^{N-n-1}-1]E_r f(n+1, m, R)\Big]\Big\}, \quad (11)$$

then the function $v(n, P, m, r) = Pf(n, m, r)$ satisfies the dynamic programming equation (10).

With just three state variables, all of which are discrete and take just finitely many values, there is reason to be hopeful that dynamic programming equation (11) can be used to solve for the optimal value function f and thus v. However, there remain questions about existence and uniqueness of a solution, how to compute a solution, and how to identify an optimal refinancing strategy. Dynamic program (11) is in the infinite horizon category, so a simple recursive procedure starting with $f(N, m, r) = 0$ will not succeed. There are several standard infinite horizon subcategories, but it is not apparent how to classify the one corresponding to (11). However, since $f(0, m(r), r) \neq k + f(0, m(r), r)$ it is straightforward (see the appendix) to obtain the following useful result:

Proposition 4.2 *The dynamic programming equation (11) is equivalent to*

$$f(n,m,r) = min\Big\{\frac{m(1+m)^{N-n}}{(1+r)[(1+m)^{N-n}-1]} + \frac{(1+m)[(1+m)^{N-n-1}-1]}{(1+r)[(1+m)^{N-n}-1]}E_r f(n+1,m,R),$$

$$k + \frac{m(r)(1+m(r))^N}{(1+r)[(1+m(r))^N-1]} + \frac{(1+m(r))[(1+m(r))^{N-1}-1]}{(1+r)[(1+m(r))^N-1]}E_r f(1,m(r),R)\Big\} \quad (12)$$

As explained in the appendix, the recursive operator defined by (12) is a contraction, and so by dynamic programming theory one concludes the following:

Theorem 4.1 *If the short rate process r is strictly positive, then dynamic programming equation (12) has a unique solution f which can be computed by either the "successive approximations" algorithm or the "policy improvement" algorithm. The corresponding minimizing argument on the right hand side of (12) gives an optimal refinancing strategy.*

Remark 4.1 Each refinancing strategy, including the optimal one, will simply be a rule that specifies for each possible value of the state vector (n,m,r) either "continue" or "refinance." The relevant dynamic programming theory and descriptions of these two and other algorithms can be found in Puterman [16].

Remark 4.2 Since dynamic programming equation (12) has a unique solution, it can be shown that with $K(P) = kP$ the dynamic programming equation (10) will also have a unique solution. It follows in this case that $f(n,m,r)$ satisfies (11) and (12) if and only if $v(n,P,m,r) = Pf(n,m,r)$ satisfies (10).

5 A Mortgage Market Equilibrium Problem

This section studies an equilibrium problem that combines the ideas of the preceding two sections. The riskless short rate r is an exogenous, time homogeneous Markov chain and, for some deterministic function $m(\cdot)$, the mortgage rate evolves according to $M_t = m(r_{t+1})$. The mortgagor is a representative agent in the mortgage market who, based upon the dynamics of the riskless short rate r and the mortgage rate function $m(\cdot)$, seeks the best refinancing strategy. Meanwhile, based upon the mortgagor's prepayment behavior, the competitive forces in the mortgage market act so the mortgage rate function $m(\cdot)$ results in mortgage contracts having initial values V equal to the loan amounts $P(0, m(r_t))$. Thus the solution of this equilibrium problem will be a pair consisting of the mortgage rate function $m(\cdot)$ together with the mortgagor's refinancing strategy, with the properties that condition (9) of Theorem 3.1 for endogenous mortgage rates is satisfied for each possible value of the short rate r and the mortgagor's refinancing strategy is optimal as per Theorem 4.1.

To compute an equilibrium solution the following "naive" algorithm can be considered. Start with an arbitrary refinancing strategy, such as "never refinance." This defines an intensity process γ_r as well as a refinance time τ_r for each possible initial value of the spot rate process r. Then using equation (9) of Theorem 3.1 one computes the endogenous mortgage rate m for each possible initial value of the spot rate process r; this defines the mortgage rate function $m(\cdot)$. Next, one takes $m(\cdot)$ and solves the dynamic programming problem of Theorem 4.1 for the optimal refinancing strategy. If this strategy is the same as the one immediately before, then stop with an equilibrium solution. If not, then proceed with another iteration by using this new strategy and (9) to compute a new mortgage rate function $m(\cdot)$, and so forth. This algorithm will be illustrated by an example in the following section.

Of course, this or any other algorithm will not converge if there does not exist a solution to the equilibrium problem. However, as explained in the appendix, it is possible to formulate a new

Markov decision chain, where the short rate process r is the single state variable, where one time period is one refinance cycle, and where the decision for each state is the choice of the stopping time representing the refinancing time. Moreover, the solution, if one exists, immediately provides a solution to the equilibrium problem. But as with the dynamic program of Theorem 4.1, the new dynamic program is known to always have a solution. Hence one has the following existence result:

Theorem 5.1 *The equilibrium problem of this section always has a solution consisting of a mortgage rate function $m(\cdot)$ and an optimal refinancing strategy for the representative mortgagor.*

Remark 5.1 The equilibrium solution is not necessarily unique, because for some state the mortgagor might be indifferent between continuation and refinancing.

6 A Numerical Example

This section provides a simple numerical example that illustrates the preceding ideas but is not realistic enough to be taken seriously as a model of actual mortgage markets. The mortgage contract has a maturity of $N = 5$ periods, and the refinancing fee is $k = 0.03$, i.e., 3%. The short rate r takes one of four values: 2%, 3%, 4%, or 5%. The corresponding Markov chain has transition probabilities

$$Q(r_{t+1} = 2\%|r_t = 2\%) = Q(r_{t+1} = 3\%|r_t = 2\%) = 1/2$$

$$Q(r_{t+1} = 2\%|r_t = 3\%) = Q(r_{t+1} = 3\%|r_t = 3\%) = Q(r_{t+1} = 4\%|r_t = 3\%) = 1/3$$

$$Q(r_{t+1} = 3\%|r_t = 4\%) = Q(r_{t+1} = 4\%|r_t = 4\%) = Q(r_{t+1} = 5\%|r_t = 4\%) = 1/3$$

$$Q(r_{t+1} = 4\%|r_t = 5\%) = Q(r_{t+1} = 5\%|r_t = 5\%) = 1/2$$

Note the resulting Markov decision chain has $4^2 \times 5 = 80$ states, although some, e.g. $(1, m(5\%), 2\%)$, are not accessible.

To compute an equilibrium solution the "naive" algorithm is started with the trading strategy where the mortgagor does not refinance, regardless of the spot rate's value at contract initiation. To solve for the four endogenous mortgage rates, use is made of equation (9) which in this case can be written as

$$\frac{1 - (1+m)^{-5}}{m} = E_{r_1}\Big[\sum_{i=1}^{5} \frac{1}{(1+r_1)\dots(1+r_i)}\Big].$$

This results in the mortgage rate function $m(\cdot)$ taking the values $2.4733\%, 3.0773\%, 3.9061\%$, and 4.5201% corresponding to $r = 2\%, 3\%, 4\%$ and 5%, respectively.

Next, the Markov decision chain is solved for the optimal refinancing strategy. It turns out that it is optimal to refinance in nine of the 80 states. But only two of these nine states are possible, namely, $(2, 4.5201\%, 2\%)$ and $(3, 3.9061\%, 2\%)$. In other words, with the mortgage rate function as above it is optimal to refinance a contract that started with $r = 5\%$ if and only if $r = 2\%$ at the end of the third period. And it is optimal to refinance a contract that started with $r = 4\%$ if and only if $r = 2\%$ at the end of the second period. But if a contract starts with either $r = 2\%$ or $r = 3\%$ then it should not be refinanced.

Since the refinancing strategy is new, the "naive" algorithm needs to proceed with at least one more iteration. For $r = 2\%$ and $r = 3\%$ the values of $m(r)$ will remain as before, but for $r = 4\%$ and $r = 5\%$ new values of endogenous mortgage rates must be computed using equation (9), which now has the form

$$1 = \frac{m}{1-(1+m)^{-5}} E_{r_1}\Big[\sum_{i=1}^{\tau\wedge 5} \frac{1}{(1+r_1)\dots(1+r_i)}\Big] + \frac{1}{1-(1+m)^{-5}} E_{r_1}\Big[\frac{1-(1+m)^{-(5-\tau)}}{(1+r_1)\dots(1+r_{\tau\wedge 5})}\Big].$$

Solving for m for the two cases $r = 4\%$ and $r = 5\%$ yields $m(4\%) = 3.9820\%$ and $m(5\%) = 4.5465\%$. The Markov decision chain is now solved with this new mortgage rate function, resulting in the same optimal refinancing strategy as with the preceding iteration. Hence the algorithm has converged to a solution of the equilibrium problem. Note the corresponding values of $f(0, m(r), r)$ in the solution of the dynamic programming equation (12) are $1.00000, 1.00000, 1.00194$, and 1.00063 for $r = 2\%, 3\%, 4\%$, and 5%, respectively. This solution and other matters will be discussed in the following section.

7 Concluding Remarks

The equilibrium problem is like a two-person, nonzero sum game where one player, the representative mortgagor, responds to given mortgage rates by choosing a refinancing strategy to minimize the expected present value of the cash flow and where the second player, the "market," responds to given mortgagor behavior by setting mortgage rates in a competitive fashion. It is interesting to note that as a result the mortgagor might "shoot himself in the foot" and force an equilibrium solution that has a higher expected present value. In other words, his myopic behavior of ignoring how his refinancing strategy affects the mortgage rates may eventually result in an expected present value that is higher than necessary. This can be seen from the numerical example. The mortgage's expected present values 1.00194 and 1.00063 at contract initiation exceed one by precisely the expected present value of the refinancing costs. If the mortgagor is content with the strategy of never refinancing, then the expected present values of the mortgages will be less, namely 1.0, in states $r = 4\%$ and $r = 5\%$, even though smaller expected present values can be obtained, provided the mortgage rate function $m(\cdot)$ remains the same.

Although the numerical example is very simple, one thing it and the more general model suggest is that, in practice, mortgagors might be too hasty to refinance just because mortgage rates have dropped. In practice mortgagors probably focus on their monthly coupon payments while ignoring the expected present value of the new mortgage. The latter might not be small enough to justify refinancing because the new mortgage will have full maturity, whereas the existing one is much closer to maturity. Since mortgage rates have dropped the riskless interest rates have probably dropped too, and so the expected present values of the more distant new coupon payments might be higher than anticipated. It is desirable for future research to examine this more carefully be producing more realistic computational results, such as for a model having 180 monthly periods and several dozen levels of the interest rate r.

An interesting issue for future research is the investigation of efficient procedures for computing a solution to the equilibrium problem. A naive algorithm was used in Section 6, but it is an open question whether this works in general. Solving directly the dynamic program associated with Theorem 5.1 might not be practical because the number of feasible actions (i.e., stopping rules) could be enormous; for example with 180 periods and 20 possible values for the riskless interest rate, the number of feasible stopping rules will be of the order of 2^{3600}. Another idea is to find conditions such that it would never be optimal to refinance when the riskless rate is at a level equal to or higher than the level when the contract originated. In this case one can immediately compute the equilibrium mortgage rate for the smallest level of the riskless rate, and so it might be possible to recursively compute the equilibrium mortgage rates one level at a time, beginning with the second smallest level of the riskless rate.

Of course actual mortgagors prepay for a variety of reasons, not just because they want to refinance the same principal. For example, some might choose to keep the principal amount unchanged while taking advantage of lower mortgage rates by paying the same monthly coupon amount as before, thereby paying off the loan more quickly than before. Others might choose to take advantage of increased property values by withdrawing equity and increasing the size of the loan. And many others prepay because, for a wide variety of circumstances, they sell their property. Given this variety of reasons and the heterogeneity of mortgagors, the equilibrium model and the concept of a representative mortgagor should probably not be taken too seriously. On the other hand, a more realistic goal for future research might be to generalize this paper's Markov decision chain model for optimal refinancing by incorporating additional mortgage prepayment reasons such as those enumerated above.

References

[1] Deng, Y., (1997), "Mortgage Termination: an Empirical Hazard Model with a Stochastic Term Structure," *J. of Real Estate and Economics* 14, 309-331.

[2] Deng, Y., J.M. Quigley, and R. Van Order, (2000), "Mortgage Termination, Heterogeneity, and the Exercise of Mortgage Options," *Econometrica* 68, 275-307.

[3] Dunn, K.B., and J.J. McConnell, (1981), "A Comparison of Alternative Models for Pricing GNMA Mortgage-Backed Securities," *J. of Finance* 36, 471-483.

[4] Dunn, K.B., and J.J. McConnell, (1981), "Valuation of Mortgage-Backed Securities," *J. of Finance* 36, 599-617.

[5] Dunn, K.B., and C.S. Spatt, (1999), "The Effect of Refinancing Costs and Market Imperfections on the Optimal Call Strategy and the Pricing of Debt Contracts," working paper, Carnegie Mellon University.

[6] Goncharov, Y., (2003), "An Intensity-Based Approach to Valuation of Mortgage Contracts Subject to Prepayment Risk," Ph.D. dissertation, University of Illinois at Chicago.

[7] Gorovoi, V., and V. Linetsky, (2005), "Instensity-based Valuation of Residential Mortgages: an Analytically Tractable Model," working paper, Northwestern University.

[8] Kalotay, A., D. Yang, and F.J. Fabozzi, (2004), "An Option-Theoretic Prepayment Model for Mortgages and Mortgage-Backed Securities," *Int. J. of Theoretical & Applied Finance*, to appear.

[9] Kariya, T., and M. Kobayashi, (2000), "Pricing Mortgage Backed Securities (MBS): a Model Describing the Burnout Effect," *Asian Pacific Financial Markets* 7, 182-204.

[10] Kariya, T., S.R. Pliska, and F. Ushiyama, (2002), "A 3-Factor Valuation Model for Mortgage Backed Securities (MBS)," working paper, Kyoto Institute of Economic Research.

[11] Kau, J.B., D.C. Keenan, and A.A. Smurov, (2004), "Reduced Form Mortgage Valuation," working paper, University of Georgia.

[12] Longstaff, F.A., (2002), "Optimal Refinancing and the Valuation of Mortgage-Backed Securities," working paper, University of California at Los Angeles.

[13] Nakagawa, H., and T. Shouda, (2004), "Valuation of Mortgage-Backed Securities Based on Unobservable Prepayment Cost Processes," working paper, Tokyo Institute of Technology.

[14] Pliska, S.R., (1997), *Introduction to Mathematical Finance: Discrete Time Models*, Blackwell, Oxford.

[15] Pliska, S.R., (2005), "Mortgage Valuation and Optimal Refinancing," *Stochastic Finance: Proceedings of a 2004 Conference in Lisbon, Portugal*, Springer-Verlag, Heidelberg.

[16] Puterman, M.L., (1994), *Markov Decision Processes: Discrete Stochastic Dynamic Programming*, J. Wiley, New York.

[17] Schwartz, E.S., and W.N. Torous, (1989), "Prepayment and the Valuation of Mortgage Backed Securities," *J. of Finance* 44, 375-392.

[18] Schwartz, E.S., and W.N. Torous, (1992), "Prepayment, Default, and the Valuation of Mortgage Pass-through Securities," *J. of Business* 65, 221-239.

[19] Stanton, R., (1995), "Rational Prepayment and the Valuation of Mortgage-Backed Securities," *Review of Financial Studies* 8, 677-708.

[20] Stanton, R., and N. Wallace, (1998), "Mortgage Choice: What's the Point?" *Real Estate Economics* 26, 173-205.

8 Appendix

Proof of Proposition 2.1: In view of (5) we have

$$V = E\Big[\sum_{i=1}^{N} \mathbf{1}_{\{\tau \geq i\}} c \prod_{j=1}^{i} \frac{1}{1+r_j}\Big] + E\Big[\sum_{i=1}^{N} \mathbf{1}_{\{\tau = i\}} P(i,m) \prod_{j=1}^{i} \frac{1}{1+r_j}\Big].$$

But

$$E\Big[\mathbf{1}_{\{\tau \geq i\}} c \prod_{j=1}^{i} \frac{1}{1+r_j}\Big] = E\Big[E\Big[\mathbf{1}_{\{\tau \geq i\}} c \prod_{j=1}^{i} \frac{1}{1+r_j}\Big|\mathcal{F}_i\Big]\Big] = E\Big[E\Big[\mathbf{1}_{\{\tau \geq i\}}\Big|\mathcal{F}_i\Big] c \prod_{j=1}^{i} \frac{1}{1+r_j}\Big]$$

$$= E\Big[\Big(\prod_{k=1}^{i-1}(1-\gamma_k)\Big) c \prod_{j=1}^{i} \frac{1}{1+r_j}\Big] = \frac{c}{1+r_1} E\Big[\prod_{j=1}^{i-1} \frac{1-\gamma_j}{1+r_{j+1}}\Big],$$

since the probability of the event $\{\tau \geq i\}$ given the time-i history (and thus the history of the short rate process up through r_{i+1}) is the $\mathcal{F}_i$ measurable random variable $\prod_{k=1}^{i-1}(1-\gamma_k)$. Similarly,

$$E\Big[\mathbf{1}_{\{\tau = i\}} P(i,m) \prod_{j=1}^{i} \frac{1}{1+r_j}\Big] = E\Big[E\Big[\mathbf{1}_{\{\tau = i\}}\Big|\mathcal{F}_i\Big] P(i,m) \prod_{j=1}^{i} \frac{1}{1+r_j}\Big]$$

$$= E\Big[\gamma_i \prod_{k=1}^{i-1}(1-\gamma_k) P(i,m) \prod_{j=1}^{i} \frac{1}{1+r_j}\Big] = \frac{1}{1+r_1} E\Big[\gamma_i P(i,m) \prod_{j=1}^{i-1} \frac{1-\gamma_j}{1+r_{j+1}}\Big].$$

Hence (6) immediately follows from these three equations.

Proof of Theorem 2.1: Using (1) to substitute for c in (6) produces

$$(1+r_1)V = E\Big[\sum_{i=1}^{N} [(1+m)P(i-1,m) - P(i,m) + \gamma_i P(i,m)] \prod_{j=2}^{i} \frac{1-\gamma_{j-1}}{1+r_j}\Big]$$

$$= \quad (1+m)P(0,m) + E\Big[(1+m)P(N-1,m)\prod_{j=2}^{N}\frac{1-\gamma_{j-1}}{1+r_j} - (1-\gamma_1)P(1,m)\Big]$$

$$+ \quad E\Big[\sum_{i=2}^{N-1}[(1+m)P(i-1,m) - (1-\gamma_i)P(i,m)]\prod_{j=2}^{i}\frac{1-\gamma_{j-1}}{1+r_j}\Big]$$

$$= (1+m)P(0,m) + E\Big[\sum_{i=2}^{N}(1+m)P(i-1,m)\prod_{j=2}^{i}\frac{1-\gamma_{j-1}}{1+r_j} - \sum_{i=1}^{N-1}(1-\gamma_i)P(i,m)\prod_{j=2}^{i}\frac{1-\gamma_{j-1}}{1+r_j}\Big]$$

$$= \quad (1+m)P(0,m) + E\Big[\sum_{i=1}^{N-1}(1-\gamma_i)\Big[\frac{(1+m)-(1+r_{i+1})}{1+r_{i+1}}\Big]P(i,m)\prod_{j=2}^{i}\frac{1-\gamma_{j-1}}{1+r_j}\Big]$$

$$= \quad (1+m)P(0,m) + E\Big[\sum_{i=1}^{N-1}(1-\gamma_i)\Big[\frac{m-r_{i+1}}{1+r_{i+1}}\Big]P(i,m)\prod_{j=2}^{i}\frac{1-\gamma_{j-1}}{1+r_j}\Big]$$

$$= \quad (1+m)P(0,m) + E\Big[\sum_{i=1}^{N-1}(m-r_{i+1})P(i,m)\prod_{j=2}^{i+1}\frac{1-\gamma_{j-1}}{1+r_j}\Big].$$

This is the same as (7).

Proof of Theorem 3.1: Equation (8) follows immediately from (7) upon replacing V by $P(0,m)$. To see (9) we first use (2) and (3) to obtain

$$P(i,m) = \frac{mP(0,m)}{1-(1+m)^{-N}}\frac{[1-(1+m)^{i-N}]}{m} = \frac{1-(1+m)^{i-N}}{1-(1+m)^{-N}}P(0,m) = \frac{(1+m)^N-(1+m)^i}{(1+m)^N-1}P(0,m).$$

Substituting this for $P(i,m)$ in (8) and then canceling the common factor $P(0,m)$ provides (9).

Proof of Proposition 4.1: Suppose f satisfies (11). Set $v(n,P,m,r) = Pf(n,m,r)$ and substitute $P^{-1}v$ for f in (11) to obtain

$$P^{-1}v(n,P,m,r) = \min\Big\{k + P^{-1}v(0,P,m(r),r),$$

$$\frac{1+m}{(1+r)[(1+m)^{N-n}-1]}\Big[m(1+m)^{N-n-1} + [(1+m)^{N-n-1}-1]E_rf(n+1,m,R)\Big]\Big\}.$$

$$= \min\Big\{k + P^{-1}v(0,P,m(r),r),$$

$$(1+r)^{-1}\Big[\frac{m(1+m)^{N-n}}{(1+m)^{N-n}-1} + \frac{(1+m)[(1+m)^{N-n-1}-1]}{(1+m)^{N-n}-1}E_rf(n+1,m,R)\Big]\Big\}.$$

Now multiply both the right and left sides by P to obtain

$$v(n,P,m,r) = \min\Big\{kP + v(0,P,m(r),r),$$

$$(1+r)^{-1}\Big[\frac{m(1+m)^{N-n}}{(1+m)^{N-n}-1}P + E_r\Big[\frac{(1+m)[(1+m)^{N-n-1}-1]}{(1+m)^{N-n}-1}Pf(n+1,m,R)\Big]\Big]\Big\}$$

$$= \min\Big\{kP + v(0,P,m(r),r),$$

$$(1+r)^{-1}\Big[\frac{m(1+m)^{N-n}}{(1+m)^{N-n}-1}P + E_r v(n+1, \frac{(1+m)[(1+m)^{N-n-1}-1]}{(1+m)^{N-n}-1}P, m, R)\Big]\Big\}.$$

Hence v satisfies (10) with $K(P) = kP$.

Proof of Proposition 4.2: Since $f(0, m(r), r) \neq k + f(0, m(r), r)$ it follows that

$$f(0, m(r), r) = \frac{1+m(r)}{(1+r)[(1+m(r))^N - 1]}\Big[m(r)(1+m(r))^{N-1} + [(1+m(r))^{N-1}-1]E_r f(1, m(r), R)\Big]\Big\}.$$

Substituting this in (11) immediately gives (12).

Proof of Theorem 4.1: It suffices to show that all the factors multiplying the expectations in (12) are strictly less than one. Indeed, for all $n \geq 1$, all $m > 0$, and all $r_t(\omega)$ we have

$$\frac{(1+m)[(1+m)^{n-1}-1]}{(1+r_t(\omega))[(1+m)^n - 1]} < \frac{1}{1+r_t(\omega)} < \frac{1}{1+r_{\min}} < 1,$$

where $r_{\min} := \min_{t,\omega} r_t(\omega) > 0$. Hence the recursive operator defined by (12), which maps the bounded function f into a Banach space of bounded functions, is a contraction. The rest follows by standard dynamic programming theory (see, for example, Puterman [16]).

Proof of Theorem 5.1: The idea is to formulate a new Markov decision chain (MDC) which is known to have a solution and which solves the equilibrium problem all at once. But to better understand this formulation, first note that the old MDC (11) involves time periods corresponding to natural units of time, states of the form (n, m, r), an "action" a to be chosen for each state that is either a = "refinance" or a = "continue," a one-period cost that is equal to k if the action is refinance and is equal to $m(1+m)^{N-n}/[(1+r)((1+m)^{N-n}-1)]$ if the action is continue, and (defective) Markov transition probabilities $p[(n', m', r')|(n, m, r), a]$ that are based on the dynamics of the short rate process r_t.

Consider instead a new MDC where the time periods correspond to refinance cycles, so there is one period for each contract until the loan is paid off. The state now is simply the short rate r at contract initiation, and the action is now a stopping time τ (this is respect to the underlying short rate Markov chain) specifying the number of coupon payments until refinancing occurs. Thus observing the initial state r, the mortgagor chooses the stopping time τ for the first contract, then refinancing in the next state with $r = r_\tau$ the mortgagor chooses a new stopping time τ for the second contract, and so forth. Of course, $\tau = N$ means there is no refinancing, as the current mortgage contract is carried to completion and the MDC is terminated.

Now consistent with our competitive markets assumption, and since choosing a stopping time is the same as choosing an intensity process γ, when the mortgagor chooses action τ in state r, then the mortgage rate m for that contract must be the solution of equation (9) (with γ corresponding to τ). In view of this, we henceforth let $m_{r\tau}$ denote this resulting, competitive mortgage rate.

The next step in the formulation of the new MDC is to specify the one period cost. Actually, this can be an expected cost; it is the expected present value under the old MDC of (i) the "coupon payments" during one refinancing cycle plus (ii) the refinancing transaction cost (if any). To see this, keep in mind that the old MDC (11) is like MDC (10) with principal $P = 1$. Then from the algebra coming from (11), the "coupon payment" at time n is

$$\frac{m_{r\tau}(1+m_{r\tau})^{N-n}}{(1+r_{1+n})[(1+m_{r\tau})^{N-n}-1]}$$

and the discount factor for time n is (note this equals zero when $n = N$)

$$\frac{(1+m_{r\tau})^n[(1+m_{r\tau})^{N-n}-1]}{(1+r_1)\dots(1+r_n)[(1+m_{r\tau})^N - 1]},$$

and so the expected present value of the coupon payments during one cycle is

$$E_r\Big[\sum_{n=0}^{\tau} \frac{m_{r\tau}(1+m_{r\tau})^N}{[(1+m_{r\tau})^N-1](1+r_1)\dots(1+r_{n+1})}\Big].$$

Note by (4) that $m_{r\tau}(1+m_{r\tau})^N/[(1+m_{r\tau})^N-1]$ is the actual coupon payment when the initial principal $P(0, m_{r\tau}) = 1$.

Similarly, multiplying the "transaction cost" k by the discount factor for $\tau = n$ gives the expected present value

$$E_r\Big[\frac{k[(1+m_{r\tau})^N-(1+m_{r\tau})^\tau]}{(1+r_1)\dots(1+r_\tau)[(1+m_{r\tau})^N-1]}\Big].$$

The one-period cost for (r, τ) is the sum of these two expressions. Note that by using (4) to substitute for c in (3) it can be seen that when the initial principal $P(0, m_{r\tau}) = 1$ then the refinancing transaction cost is $k[(1+m_{r\tau})^N-(1+m_{r\tau})^\tau]/[(1+m_{r\tau})^N-1]$.

To complete the formulation of the new MDC it remains to specify the new Markov transition probabilities. These are obtained by multiplying the indicator of the event $\{r_{\tau+1} = r'\}$ (recall r_t is predictable) by the discount factor for $\tau = n$, that is,

$$p(r'|r,\tau) = E_r\Big[\frac{[(1+m_{r\tau})^N-(1+m_{r\tau})^\tau]}{(1+r_1)\dots(1+r_\tau)[(1+m_{r\tau})^N-1]}\mathbf{1}_{\{r_{\tau+1}=r'\}}\Big].$$

It follows from the construction of the new MDC that a solution of it will in fact be a solution of the equilibrium problem. The optimal stopping rule for each initial short rate will be the mortgagor's optimal refinancing strategy, and from these one deduces the corresponding competitive mortgage rates and thus the mortgage rate function $m(r)$.

To see that this MDC always has a solution, observe by the same argument used in the proof of Theorem 4.1 that for all r and τ one has

$$\sum_{r'} p(r'|r,\tau) \le \frac{1}{1+r_{\min}} < 1,$$

where, as for Theorem 4.1, $r_{\min} := \min_{t,\omega} r_t(\omega) > 0$. Hence existence of a solution to the new MDC follows from standard dynamic programming theory.