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INVARIANCE CONDITIONS FOR RANDOM CURVATURE MODELS

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Abstract. A class of probability models is introduced with the objective of representing certain properties of the geometric optics of the human eye. Astigmatic probability laws are those in which the extreme curvature values in the anterior corneal surface, measured at circularly arranged and equally spaced locations, are displaced by an approximate 90 deg angular separation. The relationship between the symmetry invariance of these probability laws for curvature data and probability laws for the ranking permutations associated with the ordering of these data is obtained. A distinction is made between the condition in which the components of the curvature ensemble are represented as real numbers from that in which these curvatures are color-coded and take value on a finite totally ordered set. A constructive principle for astigmatic laws is outlined based on algebraic arguments for the analysis of structured data.

1. Introduction

The cornea is the main refracting surface of the human eye. Its anterior surface is approximately 1.3 cm^2 with an average curvature radius of about 7.8 mm. Typical computer algorithms for corneal curvature measurement (keratometry) are based on projecting a pattern of concentric rings of light into the anterior surface of the cornea and numerically determining the relative separation between the images of these reflected rings of light. Sampling the curvature at specific circularly equidistant points, a numerical model for the surface curvature may be obtained. The corneal curvature, κ, and its refractive index, η, contribute to determining the surface's optical refractive power $\kappa(\eta-\eta')$, where η' is the reference refractive index ($\eta'=1$) of the air. Most of the light refraction takes place at the surface of the cornea, which has refractive index $\eta=1.3376$. Light then passes through the aqueous humour ($\eta=1.336$, close to the refractive index of water) to the lens ($\eta=1.386$ to 1.406, contributing to divergence) and through the vitreous humour to the retina at the back of the eye. The standard unit of refractive optical power (or focal lengths) is the diopter (D). One diopter equals one inverse meter (m^{-1}). For example, using the standard keratometric index $\eta=1.3375$, a cornea with a curvature of 7.50 mm at 0 deg has power $(1/0.0075)\times 0.3375=45$D, whereas if the curvature at 90 deg is 7.25 mm, the power is 46.50 D. The difference between these steep (maximum) and the flat (minimum) curvatures is the amount of regular astigmatism present in the optics of the eye, which interferes with a sharp formation of the image on the retina. In the present example, the regular astigmatism is 1.50 D.

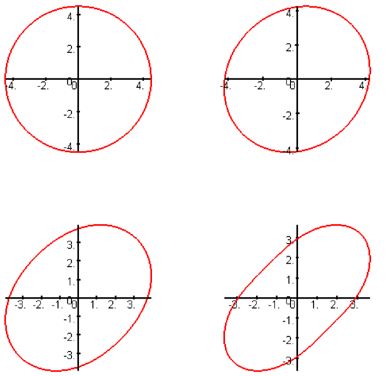
2. Astigmatic and stigmatic probability laws

The simplest geometrical representation of the corneal surface curvature corresponds to a spherical-cylindrical surface with the direction of the steep (maximum, κ_{s}) and flat (minimum, κ_{f}) curvatures oriented with a 90 deg angular separation. This is simply Euler Theorem of classical differential geometry. The resulting refractive profile is given by

$$\mathrm{y}(\theta)=(\eta-\eta')[\kappa_{\mathrm{s}}\cos^2(\theta-\alpha)+\kappa_{\mathrm{f}}\sin^2(\theta-\alpha)],\quad 0\le\theta\le 2\pi, \tag{2.1}$$

where $0\le\alpha\le\pi$ is the reference angle for the $\{\mathrm{k_s},\mathrm{k_f}\}$ orthogonal directions. Figure 1 shows the refractive profiles, in units of diopter×10, for a spherocylindrical surface ($\alpha=\pi/4$) with (a) $\kappa_{\mathrm{f}}=7.5$, $\kappa_{\mathrm{s}}=7.5$, (b)

FIGURE 1. Refractive profiles, in units of diopter×10, for a spherocylindrical surface ($\alpha = \pi/4$) with $\kappa_{\mathrm{f}} = 7.5,\ \kappa_{\mathrm{s}} = 7.5$ (upper left); $\kappa_{\mathrm{f}} = 7.5,\ \kappa_{\mathrm{s}} = 8.5$ (upper right); $\kappa_{\mathrm{f}} = 7.5,\ \kappa_{\mathrm{s}} = 10.5$ (lower left) and $\kappa_{\mathrm{f}} = 7.5,\ \kappa_{\mathrm{s}} = 15.5$ (lower right).



$\kappa_{\mathrm{f}} = 7.5,\ \kappa_{\mathrm{s}} = 8.5$, (c) $\kappa_{\mathrm{f}} = 7.5,\ \kappa_{\mathrm{s}} = 10.5$ and (d) $\kappa_{\mathrm{f}} = 7.5,\ \kappa_{\mathrm{s}} = 15.5$. The regular astigmatism is $0\mathrm{D}, 5.3\mathrm{D}, 12.8\mathrm{D}, 23.2\mathrm{D}$, respectively.

The right-angle angular separation between the extreme ordered curvatures is a characteristic of an optically astigmatic ($\kappa_{\mathrm{s}} \neq \kappa_{\mathrm{f}}$) surface. In particular, if

$$\mathrm{y}' = (\mathrm{y}(\theta_1), \ldots, \mathrm{y}(\theta_\ell))$$

is a vector of ℓ random curvatures (or refractive powers) observed at directions $\theta_{\mathrm{j}} = 2\pi \mathrm{j}/\ell$, $\mathrm{j} = 0, 1, \ldots, \ell - 1$, then the statistic of natural interest is the ordered curvatures and the corresponding induced order on the angular position θ_{j}. That is, if $\kappa_{\mathrm{f}} = \mathrm{y}(\theta_{\mathrm{f}})$ and $\kappa_{\mathrm{s}} = \mathrm{y}(\theta_{\mathrm{s}})$ then we must have $\theta_{\mathrm{s}} - \theta_{\mathrm{f}} = \pm\pi/2$. This suggests:

Definition 2.1 (Astigmatic Property)**.** The probability law $\mathcal{L}(\mathrm{y})$ of y satisfies the astigmatic property when the mean angular variation between two order statistics is $\pm\pi/2$ if and only if these are the extreme (flat and steep) order statistics.

In contrast, a spherical ($\kappa_{\mathrm{f}} = \kappa_{\mathrm{s}}$) surface would lead to a constant mean curvature and, in particular, the mean angular variation between any two ordered curvatures should be a constant. This suggests:

Definition 2.2 (Stigmatic Property)**.** The probability law $\mathcal{L}(\mathrm{y})$ of y satisfies the stigmatic property when the mean angular variation between any two order statistics is functionally independent of these order statistics.

The purpose of this paper is discussing certain invariance properties of probability laws $\mathcal{L}(\mathrm{y})$ that are consistent with the refractive properties of the underlying optical profile $\mathrm{y}(\theta)$. A characterization of the parametric symmetries in $\mathcal{L}(\mathrm{y})$ is necessary, for example, in stochastic simulation studies and in inferential hypothesis testing about the covariance structure for the components of y. In practice, the observed curvatures may be represented as points in a real vector space, or as a vector of discrete gray-scale (totally ordered) levels used to color-code a topographic corneal map. This distinction will be important later on in Sections 4 and 5. The paper is divided as follows: In the remaining of this section we will introduce the notation and structured data formulation needed to discuss the connection between the symmetries invariance in the probability law of y and the geometric properties for the law. In Section 3 we derive the first moments of the angular variation when the law of y is permutation symmetric. The case of finite-valued curvatures is discussed in Section 4, whereas a constructive rule for obtaining astigmatic laws is introduced in Section

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5. Additional comments and technical definitions are included in Section 6. The basic algebraic notions that are relevant to probability and statistics are derived from Diaconis (1988) and Eaton (1989), as well as Beckett and Diaconis (1994) and Diaconis (1989*a*).

The following notation and assumptions will be adopted in the sequence. The random variables $\mathrm{y}(\theta_{\mathrm{j}}) \equiv \mathrm{y}(\mathrm{j})$ of interest are scalar variables that take values on a totally ordered (possibly finite) set C. We identify $\mathrm{y}' = (\mathrm{y}(1), \mathrm{y}(2), \ldots, \mathrm{y}(\ell))$ as a point in the space $\mathrm{V} = \mathrm{C}^{\mathrm{L}}$ of all mappings from $\mathrm{L} = \{1, \ldots, \ell\}$ to C. The space V is described as a set of labels indexing the observable curvature values. We denote by S_ℓ the set of all permutations defined in L and recall that S_ℓ has the algebraic property of a finite group under the operation of function composition. For example, if τ is the permutation in S_3 that maps $1 \mapsto 2$, $2 \mapsto 3$ and $3 \mapsto 1$, we indicate it by writing $\tau = (123)$. Given $\tau \in \mathrm{S}_\ell$, we write $\mathrm{y}\tau^{-1}$ to indicate the composite map

$$\mathrm{y}\tau^{-1} : \mathrm{L} \xrightarrow{\tau^{-1}} \mathrm{L} \xrightarrow{\mathrm{y}} \mathrm{C}$$

of $\mathrm{y} \in \mathrm{V}$ and $\tau^{-1} \in \mathrm{S}_\ell$. Note that $\mathrm{y}\tau^{-1} \in \mathrm{V}$ and that the function $(\tau, \mathrm{y}) \in \mathrm{S}_\ell \times \mathrm{V} \to \mathrm{y}\tau^{-1} \in \mathrm{V}$ satisfies the (group action) property

$$(\sigma, (\tau, \mathrm{y})) = (\sigma\tau, \mathrm{y}) \tag{2.2}$$

for all σ, τ in S_ℓ and y in V. For simplicity of notation, however, we will write $\mathrm{y}\tau$ instead of $\mathrm{y}\tau^{-1}$ in the remaining of the paper. It is understood that the right-side composition with τ^{-1} is required to make property (2.2) work. The constructive principles described later on in Sections 4 and 5 depend directly on this property.

Because the components of y are in a totally ordered set, it is possible to order these components according to the specified total order relation ($\leq$) in C. We indicate by $\widetilde{\mathrm{y}}$ the ordered version of y and write

$$\widetilde{\tau} = \{\mathrm{y} \in \mathrm{V}; \mathrm{y}\tau = \widetilde{\mathrm{y}}\}$$

to indicate the set of all mappings y in V for which $\mathrm{y}(\tau 1) \leq \mathrm{y}(\tau 2) \leq \ldots \leq \mathrm{y}(\tau \ell)$. In particular, $\widetilde{1} = \{\mathrm{y} \in \mathrm{V}; \mathrm{y} = \widetilde{\mathrm{y}}\}$ is the set of all ordered mappings and, clearly, $\widetilde{\tau} = \widetilde{1}\tau$, for all τ in S_ℓ.

Because of the random nature of y, the permutations involved in the ordering of the components of y are also random. We refer to these random permutations as ranking permutations (see also Viana (2001)). In the sequence, we assume that the set $\widetilde{\tau}$ is $\mathcal{L}(\mathrm{y})$ – measurable, for all $\tau \in \mathrm{S}_\ell$.

Definition 2.3. If for any two distinct permutations τ and σ we have $\mathrm{P}(\widetilde{\tau} \cap \widetilde{\sigma}) = 0$ with respect to $\mathrm{P} \equiv \mathcal{L}(\mathrm{y})$, we say that

$$\mathrm{V} = \bigcup_{\tau \in \mathrm{S}_\ell} \widetilde{\tau}$$

is a stochastically disjoint partition of V.

Clearly, if $\mathrm{y} \in \widetilde{\tau} \cap \widetilde{\sigma}$ then both σ and τ are ranking permutations for y. The following proposition establishes a basic connection between the symmetries in the probability law $\mathcal{L}(\mathrm{y})$ of $\mathrm{y} \in \mathrm{V}$ (describing the observable curvatures) and the probability law $\mathcal{L}(\tau)$ of the ranking permutations $\tau \in \mathrm{S}_\ell$.

Proposition 2.1. If $\mathcal{L}(\mathrm{y}) = \mathcal{L}(\mathrm{y}\tau)$ for all $\tau \in \mathrm{S}_\ell$, and $\bigcup_\tau \widetilde{\tau}$ is a stochastically disjoint partition of V, then $\mathcal{L}(\tau)$ is the uniform (Haar) probability law in S_ℓ.

Proof. Under the stated assumptions, $\mathrm{P} = \mathcal{L}(\mathrm{y})$ induces the law $\pi = \mathcal{L}(\tau)$ in S_ℓ given by $\pi(\tau) = \mathrm{P}(\widetilde{\tau})$, so that

$$1 = \sum_{\tau \in \mathrm{S}_\ell} \mathrm{P}(\widetilde{\tau}) = \sum_{\tau \in \mathrm{S}_\ell} (\widetilde{1}\tau) = \ell! \mathrm{P}(\widetilde{1}),$$

and consequently $\pi(\tau) = \mathrm{P}(\widetilde{\tau}) = \mathrm{P}(\widetilde{1}) = 1/\ell!$ for all $\tau \in \mathrm{S}_\ell$, that is, $\mathcal{L}(\tau)$ is uniform in S_ℓ. $\square$

3. THE UNIFORM MEAN ANGULAR VARIATION

We observe the data $\mathrm{y} = (\mathrm{y}(1), \ldots, \mathrm{y}(\ell))$ according to a probability law $\mathcal{L}(\mathrm{y})$ and assume that $\bigcup_{\tau \in \mathrm{S}_\ell} \widetilde{\tau}$ is a stochastically disjoint partition of V. Let τ be the ranking permutation in S_ℓ ordering the components of y, that is, $\mathrm{y}(\tau 1) \leq \mathrm{y}(\tau 2) \leq \ldots \leq \mathrm{y}(\tau \ell)$, so that $\tau 1$ is the location of the flat (minimum) curvature and $\tau \ell$ the location of the steep (maximum) curvature. The concomitant angular displacement between the extreme curvatures is simply

$$\alpha_{\ell 1}(\tau) = |\tau\ell - \tau 1| \frac{2}{\ell}\pi,$$

so that Proposition 2.1 suggests that we consider the first moments of the angular variation under uniformly distributed ranking permutations. More precisely:

Proposition 3.1. Under the uniform probability law for $\tau \in \mathrm{S}_\ell$, we have

$$\mathrm{E}(\alpha_{\ell 1}) = \frac{2}{3}\frac{\ell+1}{\ell}\pi \quad \xrightarrow{\ell \to \infty} \quad \frac{2}{3}\pi, \quad \mathrm{Var}(\alpha_{\ell 1}) = \frac{2}{9}\frac{(\ell+1)(\ell-2)}{\ell^2}\pi^2 \quad \xrightarrow{\ell \to \infty} \quad \frac{2}{3}\pi^2.$$

Proof. Let $\phi(\tau) = |\tau\ell - \tau 1|$, so that $\phi^{-1}(1) \cup \ldots \cup \phi^{-1}(\ell - 1)$ forms a disjoint partition of S_ℓ and, under the uniform law in S_ℓ,

$$\mathrm{E}(\alpha_{\ell 1}(\tau)) = \sum_{\mathrm{k}=1}^{\ell-1} \frac{2\mathrm{k}\pi}{\ell} |\phi^{-1}(\mathrm{k})| \frac{1}{\ell!},$$

where $|\phi^{-1}(\mathrm{k})|$ is the number of permutations $\tau \in \mathrm{S}_\ell$ such that $\phi(\tau) = \mathrm{k}$. From the fact that $|\phi^{-1}(\mathrm{k})| = 2(\ell - \mathrm{k})[(\ell-2)!]$, direct computation then shows that

$$\mathrm{E}(\alpha_{\ell 1}(\tau)) = \frac{2}{3}\frac{\ell+1}{\ell}\pi, \quad \mathrm{E}(\alpha^2_{\ell 1}(\tau)) = \frac{2}{3}\frac{\ell+1}{\ell}\pi^2,$$

from which the proposed results follow. □

Similarly, we define the *natural* angular variation between the extreme values of y as

$$\mathrm{a}_{\ell 1}(\tau) = \min\{\alpha_{\ell 1}(\tau), \ell - \alpha_{\ell 1}(\tau)\},$$

so that $0 \leq \mathrm{a}(\tau) \leq \pi$. Let $\mathrm{f}(\tau) = \min\{|\tau\ell - \tau 1|, \ell - |\tau\ell - \tau 1|\}$. It then follows that for even values of $\ell \geq 3$, the number $|\mathrm{f}^{-1}(\mathrm{k})|$ of permutations in S_ℓ satisfying $\mathrm{f}(\tau) = \mathrm{k}$ is given by

$$|\mathrm{f}^{-1}(\mathrm{k})| = \begin{cases} 2\ell[(\ell-2)!], & \mathrm{k} = 1, \ldots, \frac{\ell}{2} - 1, \\ \ell[(\ell-2)!], & \mathrm{k} = \frac{\ell}{2}, \end{cases}$$

whereas, for odd values of $\ell \geq 3$,

$$|\mathrm{f}^{-1}(\mathrm{k})| = 2\ell[(\ell-2)!], \quad \mathrm{k} = 1, \ldots, \frac{\ell-1}{2}.$$

With proof similar to that of Proposition 3.1, we obtain the first moments for the natural angular variation:

Proposition 3.2. Under the uniform probability law for $\tau \in \mathrm{S}_\ell$, for odd $\ell \geq 3$, we have

$$\mathrm{E}(\mathrm{a}_{\ell 1}) = \frac{1}{2}\frac{\ell+1}{\ell}\pi \quad \xrightarrow{\ell \to \infty} \quad \frac{1}{2}\pi, \quad \mathrm{Var}(\mathrm{a}_{\ell 1}) = \frac{1}{12}\frac{(\ell+1)(\ell-3)}{\ell^2}\pi^2 \quad \xrightarrow{\ell \to \infty} \quad \frac{1}{12}\pi^2,$$

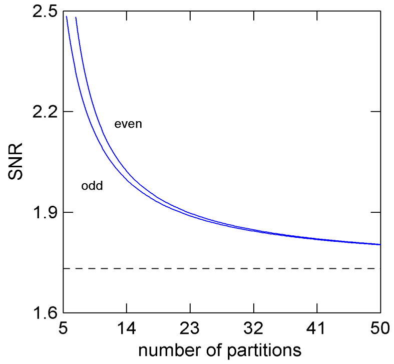
and, for even $\ell \geq 3$, we have

$$\mathrm{E}(\mathrm{a}_{\ell 1}) = \frac{1}{2}\frac{\ell}{\ell-1}\pi \quad \xrightarrow{\ell \to \infty} \quad \frac{1}{2}\pi, \quad \mathrm{Var}(\mathrm{a}_{\ell 1}) = \frac{1}{12}\frac{\ell^3 - 4\ell^2 + 8\ell^2 - 8}{\ell(\ell-1)^2}\pi^2 \quad \xrightarrow{\ell \to \infty} \quad \frac{1}{12}\pi^2.$$

Figure 2 shows the signal-to-noise ratio (inverse of the coefficient of variation) for the uniform natural angular variation, as a function of the number ℓ of partitions in the ring V. For a large number ℓ of partitions, the CV is of the order of $\sqrt{3}/3$.

The key observation to Propositions 3.1 and 3.2 is that the proofs do not depend on which two order statistics are involved in the definition of $\phi(\tau)$ or of $\mathrm{f}(\tau)$. In fact, the same results would be obtained for

FIGURE 2. Signal-to-noise ratio (SNR) for the uniform natural angular variation, as a function of the number ℓ of partitions.



the mean angular variation between the smallest and next-to-smallest values of y, say. More precisely, for all $\sigma \in \mathrm{S}_\ell$ fixed,

$$\lim_{\ell\to\infty}(\mathrm{E}[\phi(\tau)] - \mathrm{E}[\phi(\tau\sigma)]) = \lim_{\ell\to\infty}(\mathrm{E}[\mathrm{f}(\tau)] - \mathrm{E}[\mathrm{f}(\tau\sigma)]) = 0.$$

As a consequence, we learn that as long as the probability law $\mathcal{L}(\tau)$ for the ranking permutations τ in S_ℓ is uniform, the resulting probability law for the angular variation preserves the stigmatic property of $\mathcal{L}(\mathrm{y})$. In summary,

Proposition 3.3. If the probability law $\mathcal{L}(\tau)$ of τ is uniform in S_ℓ, then the probability law $\mathcal{L}(\mathrm{y})$ of y satisfies the stigmatic property.

From Proposition 2.1 we then obtain

Proposition 3.4. If $\mathcal{L}(\mathrm{y}) = \mathcal{L}(\mathrm{y}\tau)$ for all $\tau \in \mathrm{S}_\ell$, and $\bigcup_\tau \widetilde{\tau}$ is a stochastically disjoint partition of V then $\mathcal{L}(\mathrm{y})$ satisfies the stigmatic property.

For example, if $\mathcal{L}(\mathrm{y})$ is multivariate normal with mean (μ) and covariance matrix (Σ) which are invariant under the symmetry of S_ℓ, then the law of y is consistent with the stigmatic property. More precisely, if

$$\frac{1}{\ell!}\sum_{\tau\in\mathrm{S}_\ell}\rho(\tau)\mu = \mu, \quad \text{and} \quad \frac{1}{\ell!}\sum_{\tau\in\mathrm{S}_\ell}\rho(\tau)\Sigma\rho(\tau)' = \Sigma, \tag{3.1}$$

where $\rho(\tau)$ is the permutation representation (e.g., Comment 1, Section 6) of S_ℓ, then $\mathcal{L}(\mathrm{y})$ satisfies the stigmatic property. As a consequence, the construction of μ and Σ that is consistent with the astigmatic property will depend on properly choosing *smaller* sets G of symmetries from S_ℓ. This construction is described later on in Section 5.

In the next section we consider the relation between $\mathcal{L}(\tau)$ and $\mathcal{L}(\mathrm{y})$ when the assumption that $\cup_\tau\widetilde{\tau}$ is a stochastically disjoint partition of V is not obtained. This is the case of interest when C is a totally ordered finite set. In this case, if the astigmatic property is present, the fact that the angular variation between the steep and the flat observations are at right angles implies (for a smooth surface) that there are (at least) two locations along the ring with the same (tied) steep and flat observations. Consequently, the stochastically disjoint partition property (which implies that tied observables have zero probability) does not obtain. Examples of finite (totally ordered) sets are common when the actual curvature data are replaced by a gray (or color)-scale representation, e.g., corneal topography maps.

4. FINITE-VALUED CURVATURES

We observe a ℓ-dimensional vector y with components in $C = \{1, \ldots, c\}$, a totally ordered set. The label space $V = C^L$ for the observable curvatures has c^ℓ points. The *frame* $\lambda(y)$ of $y \in V$ is the c-component integer partition of ℓ describing the repeated symbols in the components of y. For example, if $c = \ell = 4$ and $C = \{a, b, c, d\}$ then there are 5 distinct frames in V, namely $4000 \equiv 4^1 0^3$, $3100 \equiv 3^1 1^1 0^2$, $2200 \equiv 2^2 0^2$, $2110 \equiv 2^1 1^2 0^1$ and $1111 \equiv 1^4$. Here are a few representatives: $\lambda(abab) = 2^2 0^2$, $\lambda(abaa) = 3^1 1^1 0^2$, $\lambda(abdc) = 1^4$, $\lambda(cccc) = 4^1 0^3$, $\lambda(aabc) = 2^1 1^2 0^1$. Also note that all points in the orbit

$$\mathcal{O}(y) = \{y\tau; \tau \in S_\ell\}$$

of y have the same ordered version $\widetilde{y}$ of y. The following examples will illustrate the construction of the decomposition of interest.

Example 4.1. Consider the simplest case in which $C = \{a, b\}$, such as with binary-colored topography maps, and $L = \{1, 2\}$. Suppose also that $a \leq b$. The space V has 4 points, namely,

$$V = \{aa,\ bb,\ ab,\ ba\},$$

and decomposes, in the natural way, according to the frames $\lambda = 20$ and $\lambda = 11$ (the two integer partitions of $\ell = 2$), that is

$$V = V_{20} \cup V_{11},$$

and each of these components decomposes into isomorphic orbits,

$$V_{20} = \mathcal{O}_{11} \cup \mathcal{O}_{12} = \{aa\} \cup \{bb\}, \quad V_{11} = \mathcal{O}_{21} = \{ab,\ ba\}.$$

For each $\tau \in S_2 = \{1, (12)\}$, we apply the definition of

$$\widetilde{\tau} = \{y \in V;\ y\tau 1 \leq y\tau 2\},$$

to obtain

$$\widetilde{1} = \{aa,\ bb,\ ab\}, \quad \widetilde{(12)} = \{aa, bb, ba\} = \widetilde{1}(12).$$

Next we introduce the probability laws $P \equiv \mathcal{L}(y)$ satisfying the permutation-symmetry $\mathcal{L}(y\tau) = \mathcal{L}(y)$ for all τ in S_ℓ. These (finitely exchangeable) probability laws have the form of convex linear combinations

$$P = f_{11}w_{11} + f_{12}w_{12} + f_{21}w_{21},$$

where

$$w_i = \begin{cases} \frac{1}{|\mathcal{O}_i|}, & y \in \mathcal{O}_i, \\ 0, & y \notin \mathcal{O}_i, \end{cases}$$

and $\sum_i f_i = 1$, $f_i \geq 0$, for $i \in \{11, 12, 21\}$, e.g., Diaconis and Freedman (1977). It then follows that

$$P(\widetilde{1}) = f_{11} + f_{12} + \frac{1}{2}f_{21} = P(\widetilde{(12)}),$$

and, because $f_{11} + f_{12} + f_{21} = 1$, the condition

$$P(\widetilde{1}) + P(\widetilde{(12)}) = 1$$

is equivalent to $f_{21} = 1$, or $f_{11} = f_{12} = 0$. In this case, P induces a well-defined probability law π in S_2, given by

$$\pi(\tau) = P(\widetilde{\tau}),$$

which is also invariant, and hence uniform. Here we see that when $\mathcal{L}(y)$ is S_2-invariant then the law $\mathcal{L}(\tau)$ of the ranking permutations τ in S_2 is uniform if and only if $V = \widetilde{1} \cup \widetilde{(12)}$ is a stochastic partition. $\square$

Example 4.2. Consider the case $c = \ell = 3$, so that C has three *levels of gray*, say. For simplicity, name these levels symbolically as $\{1, 2, 3\}$ and suppose that the total order of C leads to $1 \leq 2 \leq 3$. We observe a map $y = (y(1), y(2), y(3))$. The space V decomposes according to the frames $\lambda \in \{300, 210, 111\}$, the three integer partitions of $\ell = 3$, as

$$V = V_{300} \cup V_{210} \cup V_{111},$$

whereas each of these components decomposes into isomorphic orbits,

$$\begin{gathered}
\mathrm{V}_{300} = \mathcal{O}_{11} \cup \mathcal{O}_{12} \cup \mathcal{O}_{13}, \\
\mathrm{V}_{210} = \mathcal{O}_{21} \cup \ldots \cup \mathcal{O}_{26}, \\
\mathrm{V}_{111} = \mathcal{O}_{31}.
\end{gathered}$$

Table 4.1 summarizes the resulting orbits and classes $\widetilde{\tau}$. The plus signs indicate which mappings belong to a given class. Note that within each one of the 10 orbits there is exactly one element from the set $\widetilde{1}$ of ordered elements of V, a fact which characterizes $\widetilde{1}$, and $\widetilde{\tau}$ in general, as cross sections in V. More precisely, a subset $\Gamma \subset \mathrm{V}$ is a cross section if, for each $\mathrm{y} \in \mathrm{V}$, $\Gamma \cap \mathcal{O}(\mathrm{y})$ consists of exactly one point (see Eaton (1989, p.58) on conditions under which there is a stochastic representation of the form $\mathcal{L}(\mathrm{y}) = \mathcal{L}(\mathrm{x}\tau)$ for the law of y, where x is a random variable defined in $\widetilde{1}$ and independent of τ uniformly distributed in S_ℓ).

TABLE 4.1. The space $\mathrm{V} = \mathrm{C}^{\mathrm{L}}$ for $\mathrm{c} = \ell = 3$

$\mathcal{O}$	11	12	13	21	21	21	22	22	22	23	23	23	24	24	24	25	25	25	26	26	26	31	31	31	31	31	31
s(1)	1	2	3	1	1	2	2	2	1	1	1	3	3	3	1	2	2	3	3	3	1	1	1	3	2	2	3
s(2)	1	2	3	1	2	1	2	1	2	1	3	1	3	1	3	2	3	2	3	1	3	2	3	2	1	3	1
s(3)	1	2	3	2	1	1	1	2	2	3	1	1	1	3	3	3	2	2	1	3	3	3	2	1	3	1	2
$\widetilde{1}$	+	+	+	+					+	+					+	+					+	+					
$\widetilde{(12)}$	+	+	+	+				+		+				+		+				+					+		
$\widetilde{(13)}$	+	+	+			+	+					+	+					+	+					+			
$\widetilde{(23)}$	+	+	+		+				+		+				+		+				+		+				
$\widetilde{(123)}$	+	+	+			+		+				+		+				+		+							+
$\widetilde{(132)}$	+	+	+		+		+				+		+				+		+							+	

The invariant laws in V are convex combinations $\mathrm{P} = \sum_{\mathrm{i}} \mathrm{f_i w_i}$, where

$$\mathrm{w_i} = \begin{cases} \frac{1}{|\mathcal{O}_\mathrm{i}|}, & \mathrm{y} \in \mathcal{O}_\mathrm{i}, \\ 0, & \mathrm{y} \notin \mathcal{O}_\mathrm{i}, \end{cases}$$

and $\sum_{\mathrm{i}} \mathrm{f_i} = 1$, $\mathrm{f_i} \geq 0$, for $\mathrm{i} = 1, \ldots, 10$. More precisely,

$$\mathrm{P} = \sum_{\mathrm{i}=1}^{3} \mathrm{f_{1i} w_{1i}} + \sum_{\mathrm{j}=1}^{6} \mathrm{f_{2j} w_{2j}} + \mathrm{f_{31} w_{31}},$$

where $\mathrm{w_{1i}} = 1$ inside each orbit in $\widetilde{\lambda}_{300}$ and are zero elsewhere; $\mathrm{w_{2j}} = 1/3$ inside each orbit in $\widetilde{\lambda}_{210}$ and zero elsewhere; $\mathrm{w_{31}} = 1/6$ inside the single orbit in $\widetilde{\lambda}_{111}$ and zero elsewhere, and

$$\sum_{\mathrm{i}=1}^{3} \mathrm{f_{1i}} + \sum_{\mathrm{j}=1}^{6} \mathrm{f_{2j}} + \mathrm{f_{31}} = 1.$$

As a consequence, we obtain

$$\mathrm{P}(\widetilde{\tau}) = \sum_{\mathrm{i}=1}^{3} \mathrm{f_{1i}} + \frac{1}{3} \sum_{\mathrm{j}=1}^{6} \mathrm{f_{2j}} + \frac{1}{6} \mathrm{f_{31}}, \quad \text{for all } \tau \in \mathrm{S}_3. \tag{4.1}$$

Similarly to the previous example, the condition $\sum_{\tau \in \mathrm{S}_3} \mathrm{P}(\widetilde{\tau}) = 1$ is obtained (and hence the law of the ranking permutations is uniform in S_3) if and only if

$$\mathrm{V} = \bigcup_{\tau \in \mathrm{S}_3} \widetilde{\tau}$$

is a stochastic partition. □

Expression (4.1) reflects the fact that the space V decomposes as the sum of three orbits of size 1, corresponding to frame $3^1 0^2$, six orbits of size 3 corresponding to frame $2^1 1^1 0^1$ and one orbit of size 6 corresponding to frame 1^3, so that $|V| = 27 = 3 \times 1 + 6 \times 3 + 1 \times 6$. Standard combinatorial arguments show that, in general, there are n_λ orbits of size m_λ corresponding to frame λ, with $|V| = c^\ell = \sum_\lambda m_\lambda n_\lambda$, where

$$m_\lambda = \frac{\ell!}{(a_1!)^{m_1}(a_2!)^{m_2}\dots(a_k!)^{m_k}}, \quad n_\lambda = \frac{c!}{m_1!m_2!\dots m_k!},$$

so that (4.1) extends to

$$P(\widetilde{\tau}) = \sum_\lambda \frac{1}{m_\lambda} \sum_{j=1}^{n_\lambda} f_{j\lambda},$$

where λ varies over the (m) different frames $\lambda = a_1^{m_1} \dots a_k^{m_k}$, with $m_1 a_1 + \dots + m_k a_k = \ell$ and $m_1 + \dots + m_k = c$. This leads to

Proposition 4.1. Let $y \in C^L$ for a finite totally ordered set C and $\mathcal{L}(y)$ be S_ℓ-invariant. Then the law $\mathcal{L}(\tau)$ of the ranking permutations τ in S_ℓ is uniform if and only if $V = \bigcup_{\tau \in S_\ell} \widetilde{\tau}$ is a stochastic partition.

5. A CONSTRUCTIVE PRINCIPLE FOR ASTIGMATIC PROBABILITY MODELS

Propositions 2.1 and 4.1 suggest that the assumption of permutation invariance (as described by the symmetries of the entire group S_ℓ) will, in general, lead to stigmatic laws. This was apparent earlier on when discussing Expression (3.1). In this section, we outline the construction of astigmatic laws for a finite-valued random variable. The construction extends to the case $c = \infty$ by defining two points y and x in C^L as similar whenever they share the same frame of repeated symbols, that is, $\lambda(y) = \lambda(x)$.

We will consider a simple case in which y has $\ell = 8$ components and $C = \{a, b, c\}$, with the ordered relation $a \leq b \leq c$. The only assumption here is that C is a totally ordered (finite) set. This is the case when dealing with topographic maps colored by shades of gray, say. The case of numerically-valued data is discussed at the end of this section. The label space V has $3^8 = 6561$ points and, under the full set (S_8) of permutation symmetries, decomposes according to $|V| = \sum_\lambda m_\lambda n_\lambda$, with the corresponding components given by Table 5.1. We want to define a smaller subset $G \subseteq S_8$ of permutation symmetries which is consistent with the astigmatic property. The natural candidate, consistently with (2.1), is one which leaves invariant the components of y when these components vary according to the sequence

TABLE 5.1. Decomposition of $V = C^L$, with c = 3and $\ell = 8$, under the full set of permutation symmetries (S_8). There are n_λ orbits of size m_λ corresponding to frame λ.

λ	m_λ	n_λ	$m_\lambda n_\lambda$
800	1	3	3
710	8	6	48
620	28	6	168
611	56	3	168
530	56	6	336
521	168	6	1008
440	70	3	210
431	280	6	1680
422	420	3	1260
332	560	3	1680
total			6561

$$y(\theta_1) = a, \quad y(\theta_2) = b, \quad y(\theta_3) = c, \quad y(\theta_4) = b, \quad y(\theta_5) = a, \quad y(\theta_6) = b, \quad y(\theta_7) = c, \quad y(\theta_8) = b. \tag{5.1}$$

In this case, the flat (a) and the steep (c) curvatures are displaced by right angles with the intermediate curvature value (b) assigned to the other 4 semimeridians. The set of symmetries associated with (5.1) defines a commutative group, indicated here by $\mathrm{G} \subseteq \mathrm{S}_8$, with generators the permutations $\mathrm{h} = (2468)$, $\mathrm{t} = (15)$ and $\mathrm{v} = (37)$. The order of G is 16, with elements

$$\mathrm{G} = \{1, \mathrm{t}, \mathrm{v}, \mathrm{h}, \mathrm{h}^2, \mathrm{h}^3, \mathrm{tv}, \mathrm{th}, \mathrm{th}^2, \mathrm{th}^3, \mathrm{vh}, \mathrm{vh}^2, \mathrm{vh}^3, \mathrm{thv}, \mathrm{th}^2\mathrm{v}, \mathrm{th}^3\mathrm{v}\},$$

isomorphic to the product group $\mathrm{C}_4 \times \mathrm{C}_2 \times \mathrm{C}_2$, where C_n indicates the cyclic group of order n. Table 5.2 shows the proposed sets of generators for a selected number of partitions. As an example of an invariant law for y consider the following points $\mathrm{y} \in \mathrm{V}$,

Table 5.2. Symmetry generators for selected number of partitions (ℓ).

partitions	$\lvert\mathrm{G}\rvert$	generators
3	2	(12)
4	4	(13),(24)
5	4	(14),(23)
6	8	(1245),(36)
7	8	(16),(25),(34)
8	16	(2468),(15),(37)
9	16	(18),(27),(45),(36)

$$\begin{aligned}
\mathrm{y}_{11} &= (\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{b}, \mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{b}),\\
\mathrm{y}_{21} &= (\mathrm{a}, \mathrm{c}, \mathrm{c}, \mathrm{c}, \mathrm{a}, \mathrm{c}, \mathrm{c}, \mathrm{c}),\\
\mathrm{y}_{22} &= (\mathrm{a}, \mathrm{a}, \mathrm{c}, \mathrm{a}, \mathrm{a}, \mathrm{a}, \mathrm{c}, \mathrm{a}),\\
\mathrm{y}_{31} &= (\mathrm{a}, \mathrm{a}, \mathrm{a}, \mathrm{a}, \mathrm{a}, \mathrm{a}, \mathrm{a}, \mathrm{a}),\\
\mathrm{y}_{32} &= (\mathrm{b}, \mathrm{b}, \mathrm{b}, \mathrm{b}, \mathrm{b}, \mathrm{b}, \mathrm{b}, \mathrm{b}),\\
\mathrm{y}_{33} &= (\mathrm{c}, \mathrm{c}, \mathrm{c}, \mathrm{c}, \mathrm{c}, \mathrm{c}, \mathrm{c}, \mathrm{c}),
\end{aligned}$$

each of which is fixed by all permutations in G, so that these points define single-element orbits in V. Next, assign G-invariant probabilities to $\mathrm{P} = \mathcal{L}(\mathrm{y})$ according to convex combinations $\mathrm{P} = \sum_\mathrm{i} \mathrm{f}_\mathrm{i}\mathrm{w}_\mathrm{i}$, where

$$\mathrm{w}_\mathrm{i} = \begin{cases} \frac{1}{|\mathcal{O}_\mathrm{i}|}, & \mathrm{y} \in \mathcal{O}_\mathrm{i}, \\ 0, & \mathrm{y} \notin \mathcal{O}_\mathrm{i}, \end{cases}$$

and $\sum_\mathrm{i} \mathrm{f}_\mathrm{i} = 1$, $\mathrm{f}_\mathrm{i} \geq 0$, for $\mathrm{i} \in \{11, 21, 22, 31, 32, 33\}$. Note that each orbit $\mathcal{O}_\mathrm{i}$ consists of exactly one element. For example, take

$$\mathrm{f}_\mathrm{i} = \begin{cases} \frac{16}{20}, & \mathrm{i} \in \{11\}, \\ \frac{3}{40}, & \mathrm{i} \in \{21, 22\}, \\ \frac{1}{60}, & \mathrm{i} \in \{31, 32, 33\}. \end{cases}$$

Direct computation then shows that the mean natural angular variation between flat and steep curvatures, under $\mathcal{L}(\mathrm{y})$, is $0.93\pi/2$, with a standard deviation of $0.178\pi/2$. In contrast, the mean natural angular variation between flat and “next to flat” curvatures is $0.5375\pi/2$, thus reflecting the approximately astigmatic property of $\mathcal{L}(\mathrm{y})$. We conclude this section with comments on constructing the astigmatic laws for numerically-valued random curvatures.

5.1. **Numerically-valued astigmatic laws.** Here we suppose that C^{L} is a (real) vector space, so that the usual operations with vectors are defined. In particular, the vector of means $\mu = \mathrm{E}(\mathrm{y}) \in \mathrm{V}$ of y and the covariance matrix Σ of y, based on $\mathcal{L}(\mathrm{y})$ are well-defined. We retain the same set G of symmetries defined earlier in this section. Of interest now is the invariant parametric structure to be imposed into μ and Σ, consistent with the astigmatic property. These symmetry conditions are obtained, in general, by linearly representing the symmetries of interest, that is, through the permutation representation

$$\tau \in \mathrm{G} \rightarrow \rho(\tau) \in \mathrm{GL}(8, \mathbb{R})$$

and calculating

$$\mathcal{P}_1 = \frac{1}{16} \sum_{\tau \in \mathrm{G}} \rho(\tau) = \frac{1}{16} \begin{bmatrix} 8 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 8 & 0 & 0 & 0 & 8 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 8 & 0 & 0 & 0 & 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \\ 0 & 0 & 8 & 0 & 0 & 0 & 8 & 0 \\ 0 & 4 & 0 & 4 & 0 & 4 & 0 & 4 \end{bmatrix}.$$

The equation $\mathcal{P}_1 \mu = \mu$ then implies that μ should have the form $\mu = (\mu_{\mathrm{a}}, \mu_{\mathrm{b}}, \mu_{\mathrm{c}}, \mu_{\mathrm{b}}, \mu_{\mathrm{a}}, \mu_{\mathrm{b}}, \mu_{\mathrm{c}}, \mu_{\mathrm{b}})$, with the additional astigmatic condition $\mu_{\mathrm{a}} < \mu_{\mathrm{b}} < \mu_{\mathrm{c}}$ imposed. Similarly, to determine the pattern of the parametric structure of the covariance matrix Σ, we make use of the fact that Σ has the symmetry of $\mathrm{G} \subseteq \mathrm{S}_\ell$ if and only if Σ commutes will all the permutation matrices representing G. Equivalently, Σ should satisfy the condition

$$\Sigma = \sum_{\tau \in \mathrm{G}} \rho(\tau) \Sigma \rho(\tau)'.$$

This same condition appears in the definition of the corresponding likelihood ratio test statistic, e.g., Gao and Marden (2001). The solution is the class of all $\Sigma > 0$ patterned according to

$$\Sigma = \begin{bmatrix} \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{B} & \mathrm{D} & \mathrm{B} & \mathrm{C} & \mathrm{B} \\ \mathrm{B} & \mathrm{E} & \mathrm{F} & \mathrm{G} & \mathrm{B} & \mathrm{J} & \mathrm{F} & \mathrm{G} \\ \mathrm{C} & \mathrm{F} & \mathrm{H} & \mathrm{F} & \mathrm{C} & \mathrm{F} & \mathrm{I} & \mathrm{F} \\ \mathrm{B} & \mathrm{G} & \mathrm{F} & \mathrm{E} & \mathrm{B} & \mathrm{G} & \mathrm{F} & \mathrm{J} \\ \mathrm{D} & \mathrm{B} & \mathrm{C} & \mathrm{B} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{B} \\ \mathrm{B} & \mathrm{J} & \mathrm{F} & \mathrm{G} & \mathrm{B} & \mathrm{E} & \mathrm{F} & \mathrm{G} \\ \mathrm{C} & \mathrm{F} & \mathrm{I} & \mathrm{F} & \mathrm{C} & \mathrm{F} & \mathrm{H} & \mathrm{F} \\ \mathrm{B} & \mathrm{G} & \mathrm{F} & \mathrm{J} & \mathrm{B} & \mathrm{G} & \mathrm{F} & \mathrm{E} \end{bmatrix},$$

described by at most 10 distinct covariance parameters $\{\mathrm{A}, \mathrm{B}, \ldots, \mathrm{J}\}$. Diaconis (1989*b*) provides a discussion of how similar patterns arising from certain group operations can be obtained.

6. SUMMARY AND ADDITIONAL COMMENTS

We have introduced a class of probability laws suggested by the geometric optics of the human eye. These models are concerned with the representation of random corneal curvature measurements, y, indexed by concentric *equally-spaced* locations $\mathrm{V} = \{\theta_1, \ldots, \theta_\ell\}$. In the present paper we considered the effect that the symmetries imposed on the probability law of y have on the probability law for the ranking permutations associated with the ordering of the observed curvatures, as well as on the resulting geometric optics (astigmatic or stigmatic) that is consistent with those symmetries. That is, the characteristic right-angle displacement between the induced extreme order statistics in V.

6.1. **Comments.**

1. The permutation matrix $\rho(\tau)$ associated with a permutation τ is the matrix representing the linear transformation that maps the canonical basis $\{\mathrm{e}_1, \ldots, \mathrm{e}_\ell\}$ of $\mathbb{R}^\ell$ into the basis $\{\mathrm{e}_{\tau(1)}, \ldots \mathrm{e}_{\tau(\ell)}\}$. The correspondence $\tau \in \mathrm{G} \to \rho(\tau) \in \mathrm{GL}(\ell, \mathbb{R})$ is defined in G with values in the space $\mathrm{GL}(\ell, \mathbb{R})$ of invertible square matrices of order ℓ over the field of complex numbers, with the homomorphic property $\rho(\tau\sigma) = \rho(\tau)\rho(\sigma)$ for all τ, σ in G. This characterizes ρ as a ℓ-dimensional linear representation of G (the permutation representation).
2. The factor $|\tau\ell - \tau 1|$ present in the angular variation can be expressed as $\mathrm{d}'\rho(\tau)\mathrm{r}$, where $\mathrm{d}' = (-1, 0, \ldots, 0, 1)$, $\mathrm{r}' = (1, 2, \ldots, \ell)$ and ρ is the linear (permutation) representation of S_ℓ. Consequently,
$$|\tau\ell - \tau 1|^2 = \mathrm{d}'\rho(\tau)(\mathrm{rr}')\rho(\tau)'\mathrm{d},$$
and the derivation of the (uniform) mean squared angular variation can be obtained from the fact, as shown in Viana (2001), that
$$\frac{1}{\ell!}\sum_{\tau \in \mathrm{S}_\ell} \rho(\tau)\mathrm{H}\rho(\tau)' = \mathrm{v}_0\frac{\mathrm{ee}'}{\mathrm{n}} + \mathrm{v}_1(\mathrm{I} - \frac{\mathrm{ee}'}{\mathrm{n}}),$$
where $\mathrm{E}' = (1, \ldots, 1)$ with ℓ components, $\mathrm{v}_0 = \mathrm{e}'\mathrm{He}/\mathrm{n}$ is the sum of the components of H, and $\mathrm{v}_0 + (\mathrm{n}-1)\mathrm{v}_1 = \mathrm{tr}\ \mathrm{H}$, for any given (real or complex) matrix H of dimension ℓ (in this case $\mathrm{H} = \mathrm{rr}'$).
3. Average linear ranks. The derivation of the mean angular variation between flat and steep curvatures of Section 5 makes use of the usual averaging of ranks. The notion of linear representation of order statistics is useful to describe the averaging process and linearly represent the derivation of the corresponding ranks. Here is the outline: Note from Table 4.1 that the map $\mathrm{y} = (1, 1, 2)$ belongs to both $\widetilde{1}$ and $\widetilde{(12)}$. To indicate this, let
$$\widetilde{\mathrm{y}} = \{\tau; \mathrm{y} \in \widetilde{\tau}\}.$$
It then follows that the mean linear rank $\mathrm{R(y)}$ of y, under a uniform law in S_ℓ, is given by
$$\mathrm{R(y)} = \frac{1}{|\widetilde{\mathrm{y}}|}\sum_{\widetilde{\mathrm{y}}} \rho(\tau)'\mathrm{r},$$
where $\mathrm{r}' = (1, 2, \ldots, \ell)$ and ρ is the permutation representation of S_ℓ. To illustrate, consider $\mathrm{y} = (1, 1, 2)$, so that $\widetilde{\mathrm{y}} = \{1, (12)\}$. We have
$$\frac{1}{2}[\rho(1)' + \rho((12))'] = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 3 \end{bmatrix}.$$
For $\mathrm{y} = (1, 2, 1)$, similarly, we have $\widetilde{\mathrm{y}} = \{(23), (132)\}$, and
$$\frac{1}{2}[\rho((23))' + \rho((132))'] = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 3 \\ 1.5 \end{bmatrix}.$$
4. Definition 2.2 has also a stochastic interpretation, in that $\mathcal{L}(\mathrm{y})$ is permutation symmetric then the order statistics and the corresponding induced order statistics on $\mathrm{V} = \{\theta_1, \ldots, \theta_\ell\}$ (the linear ranks) are statistically independent.

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