CORRELATION ANALYSIS OF ORDERED OBSERVATIONS FROM A BLOCK-EQUICORRELATED MULTIVARIATE NORMAL DISTRIBUTION

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Abstract. Inferential procedures are developed for correlations between the order statistics from two normally distributed random vectors that are permutation symmetric. These models are used in the analysis of intra-ocular pressure data.

Contents

1. Introduction

Multivariate extreme observations are often present in vision research. Two common examples are the determination of corneal astigmatism and the assessment of extreme visual acuities. Corneal astigmatism is obtained by keratometry, which is the measurement of corneal curvature of a small area using a sample of four reflected points of light along an annulus 3 to 4 mm in diameter, centered about the line of sight. Computer-analyzed topography is similar to keratometry in that the relative separation of reflected points of light are used to calculate the curvature (Y) of the measured surface. Using a pattern of concentric rings and sampling at imaginary ring-semimeridiam intersections, a numerical model of the measured surface may be obtained. At any specified concentric ring, the extreme curvatures represent

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the steepest and flattest curvatures and their positive difference α is related to the amount of astigmatism at that ring or refractive aperture. Given p curvatures ($\mathbf{Y}$) corresponding to a subregion $\mathcal{G}$, let $\mathcal{Y} = (\mathrm{Y}_{(1)}, \mathrm{Y}_{(2)}, \ldots, \mathrm{Y}_{(\mathrm{p})})$ denote the ordered curvatures ranked from the smallest, flattest curvature $\mathrm{Y}_{(1)}$ to the largest, steepest curvature $\mathrm{Y}_{(\mathrm{p})}$. Then

$$\alpha = \mathrm{Y}_{(\mathrm{p})} - \mathrm{Y}_{(1)}$$

is the curvature range [for further details see Viana, Olkin and McMahon (1993)].

Extreme observations between fellow eyes are often used to describe visual acuity. Normally, a single joint measure $\mathrm{Y}_1, \mathrm{Y}_2$ of visual acuity is made in each eye, together with one or more covariates $\mathbf{X}$, such as the subject's age or physical condition. Because visual acuities generally are unequal, of interest are not the measures $\mathrm{Y}_1, \mathrm{Y}_2$ but rather the extreme visual acuities, the "best" acuity $\mathrm{Y}_{(1)}$ and the "worst" acuity $\mathrm{Y}_{(2)}$. Consequently, there is interest in making inferences on the covariance structure of $\mathcal{Y}, \mathrm{X}$. Such models have been considered recently by Olkin and Viana (1995). However, when $\mathcal{Y}_\mathrm{i}$ represents a vector of ordered visual acuities

$$\mathrm{Y}_{\mathrm{i}(1)} \leq \mathrm{Y}_{\mathrm{i}(2)}, \ldots, \mathrm{Y}_{\mathrm{i}(\mathrm{p})}$$

at different time-points $\mathrm{i} = 1, 2$, e.g., pre-treatment and post-treatment, then the structure of interest becomes

$$\Psi = \begin{bmatrix} \mathrm{Cov}(\mathcal{Y}_1) & \mathrm{Cov}(\mathcal{Y}_1, \mathcal{Y}_2) \\ \mathrm{Cov}(\mathcal{Y}_2, \mathcal{Y}_1) & \mathrm{Cov}(\mathcal{Y}_2) \end{bmatrix} = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{21} & \Psi_{22} \end{bmatrix}. \tag{1.1}$$

In general, the correlations of interest are

$$\eta_{\mathrm{ii,jk}} = \mathrm{Cor}(\mathrm{Y}_{\mathrm{i(j)}}, \mathrm{Y}_{\mathrm{i(k)}}), \quad \eta_{12,\mathrm{jk}} = \mathrm{Cor}(\mathrm{Y}_{1(\mathrm{j})}, \mathrm{Y}_{2(\mathrm{k})}).$$

Note that the same structure is required to express, for example, the correlation between astigmatism of fellow eyes or at two different time-points. In the present work we consider the covariance structure described by Ψ when

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} = \begin{bmatrix} \mathrm{Cov}(\mathbf{Y}_1) & \mathrm{Cov}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \mathrm{Cov}(\mathbf{Y}_2, \mathbf{Y}_1) & \mathrm{Cov}(\mathbf{Y}_2) \end{bmatrix}$$

has a block-equicorrelated structure. In the bivariate case, for example, Σ is determined by

$$\Sigma_{11} = \sigma_1^2 \begin{bmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{bmatrix}, \ \Sigma_{22} = \sigma_2^2 \begin{bmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{bmatrix}, \ \Sigma_{12} = \sigma_1\sigma_2 \begin{bmatrix} \gamma & \phi \\ \phi & \gamma \end{bmatrix},$$

which assumes that (pre-test) measurements $\mathbf{Y}_1$ on fellow eyes have a common variance σ_1^2 and a common correlation ρ_1, that corresponding (post-test) measurements $\mathbf{Y}_2$ on fellow eyes have a common variance σ_2^2 and a common correlation ρ_2, that corresponding (same-eye) measurements have a common correlation γ between time-points, whereas adjacent (contralateral-eye) measurements have a common correlation ϕ between time-points.

The following matrix notation will be used. Denote by $\mathbf{v}_{\mathrm{ij}}$ the $\mathrm{p_i} \times \mathrm{p_j}$ matrix with all entries equal to 1, by $\mathbf{I}_{\mathrm{ij}}$ the $\mathrm{p_i} \times \mathrm{p_j}$ ($\mathrm{p_i} = \mathrm{p_j}$) matrix with ones along the main diagonal entries and zeros in the remaining entries, and let

$$\mathrm{J}_{\mathrm{ij}}(\mathrm{g}, \mathrm{h}) = \sigma_\mathrm{i}\sigma_\mathrm{j}[\mathrm{h}\mathbf{v}_{\mathrm{ij}} + (\mathrm{g} - \mathrm{h})\mathbf{I}_{\mathrm{ij}}].$$

Then $\mathrm{J_{ii}(g,h)}$ indicates the matrix of dimension $\mathrm{p_i \times p_i}$ with common main diagonal entries equal to σ_i^2 g and common off diagonal entries equal to σ_i^2 h, whereas $\mathrm{J_{ij}(g,g)}$ indicates the $\mathrm{p_i \times p_j}$ matrix with all entries equal to $\sigma_\mathrm{i}\sigma_\mathrm{j}$ g.

In this paper we consider permutation-symmetric random vectors $\mathbf{Y}_1$ of dimension $\mathrm{p_1}$ and $\mathbf{Y}_2$ of dimension $\mathrm{p_2}$ jointly normally distributed with covariance structure Σ

$$\Sigma = \begin{bmatrix} \mathrm{J}_{11}(1,\rho_1) & \mathrm{J}_{12}(\gamma,\phi) \\ \mathrm{J}_{21}(\gamma,\phi) & \mathrm{J}_{22}(1,\rho_2) \end{bmatrix}, \tag{1.2}$$

subject to the restricted $\phi = \gamma$. Under this model we show that

$$\Psi = \begin{bmatrix} \Sigma_{11}\mathcal{C}_1 & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22}\mathcal{C}_2 \end{bmatrix},$$

where $\mathcal{C}_\mathrm{i}$ is the covariance matrix among $\mathrm{p_i}$ ordered independent standard normal variates.

The covariance matrix Ψ under model (1.2) restricted by $\gamma = \phi$ is derived for multivariate distributions commonly generated through certain affine transformations $\mathbf{Y} = \mathbf{m} + \mathbf{TU}$ [e.g., Guttman (1954), Tong (1990, p. 183)]. In Section 4 we review the inferential results for the covariance matrix Σ. The corresponding large-sample distributions for the maximum likelihood estimators associated with Σ and Ψ are derived in Section 5. A numerical application is discussed in Section 6 and selected derivations are outlined in Section 8.

2. The covariance structure of ordered affine observations

Given a $\mathrm{p \times m}$ real non-negative constant matrix $\mathbf{T}$ and a random vector $\mathbf{U}$ with $\mathrm{m \geq p}$ components, denote by

$$\mathbf{Y} = \mathbf{m} + \mathbf{TU}, \quad \mathbf{m} \in \mathbb{R}^\mathrm{p},$$

the multivariate (affine) transformation with generating random vector $\mathbf{U}$. It is assumed that the components of $\mathbf{U}$ are jointly independent with $\mathrm{E}(\mathbf{U}) = \mathbf{0}$ and variance 1.

Example 2.1. If the distribution of $\mathbf{Y}$ is multivariate normal of dimension p with vector of means $\mathbf{m}$ and common non-negative correlation ρ, then $\mathbf{Y} = \mathbf{m} + \mathbf{TU}$, with generating vector $\mathbf{U} \sim \mathrm{N}(0,\mathbf{I})$ of dimension $\mathrm{p}+1$ and transformation matrix $\mathrm{T} = [\sqrt{\rho}\mathbf{e}, \quad \sqrt{1-\rho}\,\mathbf{I}]$, where $\mathbf{e}$ has dimension p and $\mathbf{I}$ is the identity matrix also of dimension p.

Example 2.2. If the distribution of $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2)$ of dimension $\mathrm{q = p_1 + p_2}$ is multivariate normal with vector of means $(\mathbf{m}_1, \mathbf{m}_2)$ and block-equicorrelated covariance structure (1.2) restricted to $\phi = \gamma$, then $\mathbf{Y} = \mathbf{m} + \mathbf{TU}$ with generating vector $\mathbf{U} \sim \mathrm{N_{q+1}}(0,\mathbf{I})$ and transformation matrix

$$\mathbf{T} = \begin{bmatrix} \mathrm{a}_0\mathbf{e}_1 & \mathrm{a}\mathbf{I}_1 & \alpha\mathbf{v}_{12} \\ \mathrm{b}_0\mathbf{e}_2 & \beta\mathbf{v}_{12}^{'} & \mathrm{b}\mathbf{I}_2 \end{bmatrix}, \tag{2.1}$$

where $\mathbf{e}_\mathrm{i}$ is the $\mathrm{p_i} \times 1$ vector of ones, for appropriate choices of $\mathrm{a_0}, \mathrm{a}, \alpha, \mathrm{b_0}, \mathrm{b}, \beta \in \mathbb{R}$. In fact, to see this, note that if

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{m}_1 \\ \mathbf{m}_2 \end{bmatrix} + \mathbf{T} \begin{bmatrix} \mathrm{U}_0 \\ \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix}, \tag{2.2}$$

then $\mathbf{Y} \sim \mathrm{N}(0, \Sigma)$, with

$$\begin{aligned}
\Sigma_{11} &= \mathrm{J}_{11}(\mathrm{a}_0^2 + \mathrm{p}_2\alpha^2 + \mathrm{a}^2, \mathrm{a}_0^2 + \mathrm{p}_2\alpha^2), \\
\Sigma_{12} &= \mathrm{J}_{12}(\mathrm{a_0 b_0} + \mathrm{a}\beta + \alpha\mathrm{b}, \mathrm{a_0 b_0} + \mathrm{a}\beta + \alpha\mathrm{b}), \\
\Sigma_{22} &= \mathrm{J}_{22}(\mathrm{b}_0^2 + \mathrm{p}_1\beta^2 + \mathrm{b}^2, \mathrm{b}_0^2 + \mathrm{p}_1\beta^2).
\end{aligned}$$

Given $\sigma_1, \sigma_2, \rho_1, \rho_2, \gamma$, set

$$\mathrm{a}^2 = \sigma_1^2(1 - \rho_1), \quad \mathrm{b}^2 = \sigma_2^2(1 - \rho_2),$$

and solve

$$\sqrt{\sigma_1^2\rho_1 - \mathrm{p}_2\alpha^2}\sqrt{\sigma_2^2\rho_2 - \mathrm{p}_1\beta^2} + \beta\sigma_1\sqrt{1 - \rho_1} + \alpha\sigma_2\sqrt{1 - \rho_2} = \sigma_1\sigma_2\gamma$$

for α and β, to yeld

$$\mathrm{a}_0 = \sqrt{\sigma_1^2\rho_1 - \mathrm{p}_2\alpha^2}, \quad \mathrm{b}_0 = \sqrt{\sigma_2^2\rho_2 - \mathrm{p}_1\beta^2}.$$

Proposition 2.1. If $\mathbf{Y} = \mathbf{TU}$ with transformation matrix $\mathbf{T} = [\mathrm{a}_0\mathbf{e}, \;\; \mathrm{a}\mathbf{I}]$, $\mathrm{a} \neq 0$, and arbitrary generating vector $\mathbf{U} = (\mathrm{U}_0, \mathbf{U}_1)$, then

$$\mathrm{Cov}(\mathcal{Y}) = \mathrm{a}_0^2\mathbf{v}_{11} + \mathrm{a}^2\mathrm{Cov}(\mathcal{U}_1),$$

where $\mathrm{Cov}(\mathcal{U}_1)$ is the covariance matrix of the ordered components

$$\mathcal{U}_1 = (\mathrm{U}_{(1)}, \ldots, \mathrm{U}_{(\mathrm{p})})$$

of $\mathbf{U}_1$, when $\mathrm{a} > 0$, and of $\mathcal{U}_1 = (\mathrm{U}_{(\mathrm{p})}, \ldots, \mathrm{U}_{(1)})$ when $\mathrm{a} < 0$.

Proof. Because

$$\mathcal{Y} = \mathbf{T} \begin{bmatrix} \mathrm{U}_0 \\ \mathcal{U}_1 \end{bmatrix}, \tag{2.3}$$

where $\mathcal{U}_1$ is defined as above according to the sign of a, we have

$$\mathrm{Cov}(\mathcal{Y}) = \mathbf{T} \begin{bmatrix} 1 & \mathrm{Cov}(\mathrm{U}_0, \mathcal{U}_1)' \\ \mathrm{Cov}(\mathrm{U}_0, \mathcal{U}_1) & \mathrm{Cov}(\mathcal{U}_1) \end{bmatrix} \mathbf{T}'.$$

Moreover, since the components of $(\mathrm{U}_0, \mathbf{U}_1)$ are independent, then U_0 and $\mathcal{U}_1$ are independent and hence

$$\mathrm{Cov}(\mathcal{Y}) = \mathbf{T} \begin{bmatrix} 1 & \mathbf{0}' \\ \mathbf{0} & \mathrm{Cov}(\mathcal{U}_1) \end{bmatrix} \mathbf{T}' = \mathrm{a}_0^2\mathbf{v}_{11} + \mathrm{a}^2\mathrm{Cov}(\mathcal{U}_1).$$

$\square$

Note that if, in addition to the conditions of proposition 2.1, $\mathrm{Cov}(\mathcal{U}_1)$ is proportional to a doubly stochastic matrix, then so is $\mathrm{Cov}(\mathcal{Y})$. Also note that Proposition 2.1 holds when the mean of $\mathbf{Y}$ is a permutation-symmetric vector $\mathbf{m}$.

Proposition 2.2. If

$$\mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathrm{U}_0 \\ \mathbf{U}_1 \\ \mathbf{U}_2 \end{bmatrix} \equiv \mathbf{T}\mathbf{U},$$

with generating vector $\mathbf{U}$ and transformation matrix (2.1) then

$$\begin{aligned} \mathrm{Cov}(\mathcal{Y}_1) &= \mathrm{a}_0^2\mathbf{v}_{12} + \mathrm{a}^2\mathrm{Cov}(\mathcal{U}_1) + \alpha^2\mathbf{v}_{12}\mathrm{Cov}(\mathcal{U}_2)\mathbf{v}_{12}^{'}, \\ \mathrm{Cov}(\mathcal{Y}_1, \mathcal{Y}_2) &= \mathrm{a}_0\mathrm{b}_0\mathbf{v}_{12} + \mathrm{a}\beta\mathrm{Cov}(\mathcal{U}_1)\mathbf{v}_{12} + \alpha\mathrm{b}\mathbf{v}_{12}\mathrm{Cov}(\mathcal{U}_2), \\ \mathrm{Cov}(\mathcal{Y}_2) &= \mathrm{b}_0^2\mathbf{v}_{22} + \beta^2\mathbf{v}_{12}^{'}\mathrm{Cov}(\mathcal{U}_1)\mathbf{v}_{12} + \mathrm{b}^2\mathrm{Cov}(\mathcal{U}_2), \end{aligned}$$

where $\mathrm{Cov}(\mathcal{U}_\mathrm{i})$ is the covariance matrix of the ordered components

$$\mathcal{U}_\mathrm{i} = (\mathrm{U}_{(1)}, \ldots, \mathrm{U}_{(\mathrm{p}_\mathrm{i})})$$

of $\mathbf{U}_\mathrm{i}$, $\mathrm{i} = 1, 2$ and the ordered vectors $\mathcal{U}_1, \mathcal{U}_2$ of dimension p_1 and p_2, respectively, are defined according to the sign of a and b respectively.

Proof. Because $\mathbf{e}_\mathrm{i}^{'}\mathrm{U}_\mathrm{i} = \mathbf{e}_\mathrm{i}^{'}\mathcal{U}_\mathrm{i}$, note that

$$\begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} \mathrm{U}_0 \\ \mathcal{U}_1 \\ \mathcal{U}_2 \end{bmatrix}, \tag{2.4}$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are defined according to the sign of a and b, respectively. Moreover, since the components of $(\mathrm{U}_0, \mathbf{U}_1, \mathbf{U}_2)$ are independent, then U_0, $\mathcal{U}_1$ and $\mathcal{U}_2$ are also independent, and hence

$$\mathrm{Cov}(\begin{bmatrix} \mathcal{Y}_1 \\ \mathcal{Y}_2 \end{bmatrix}) = \mathbf{T} \begin{bmatrix} 1 & \mathbf{0}^{'} & \mathbf{0}^{'} \\ \mathbf{0} & \mathrm{Cov}(\mathcal{U}_1) & \mathbf{0}^{'} \\ \mathbf{0} & \mathbf{0} & \mathrm{Cov}(\mathcal{U}_2) \end{bmatrix} \mathbf{T}'. \tag{2.5}$$

Direct evaluation of (2.5) leads to the proposed covariance matrices. □

Corollary 2.1. Under the conditions of Proposition 2.2, if $\mathrm{Cov}(\mathcal{U}_\mathrm{i})$ is stochastic (all row and column sums equal to 1), then

$$\begin{aligned} \mathrm{Cov}(\mathcal{Y}_1) &= \Sigma_{11}\mathrm{Cov}(\mathcal{U}_1), \\ \mathrm{Cov}(\mathcal{Y}_1, \mathcal{Y}_2) &= \Sigma_{12}, \\ \mathrm{Cov}(\mathcal{Y}_2) &= \Sigma_{22}\mathrm{Cov}(\mathcal{U}_2). \end{aligned}$$

Because $\mathrm{Cov}(\mathcal{U}_\mathrm{i})$ is symmetric and stochastic (and thereby is symmetric and doubly stochastic) and Σ_{ii} is a linear combination of $\mathbf{v}_{\mathrm{ii}}$ and $\mathbf{I}$, the matrices $\mathrm{Cov}(\mathcal{U}_\mathrm{i})$ and Σ_{ii} commute and consequently their product is also symmetric. Also note that Proposition 2.2 holds when the means of $\mathbf{Y}_\mathrm{i}$ are permutation-symmetric vectors $\mathbf{m}_\mathrm{i}$, $\mathrm{i} = 1, 2$.

Corollary 2.2. If the distribution of $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2)$ of dimension $\mathrm{q} = \mathrm{p}_1 + \mathrm{p}_2$ is multivariate normal with permutation-symmetric means $\mathbf{m}_1$ and $\mathbf{m}_2$, and block-equicorrelated covariance structure (1.2) with $\phi = \gamma$, then

$$\begin{aligned} \mathrm{Cov}(\mathcal{Y}_\mathrm{i}) &= \Sigma_{\mathrm{ii}}\mathcal{C}_\mathrm{i}, \quad \mathrm{i} = 1, 2 \\ \mathrm{Cov}(\mathcal{Y}_1, \mathcal{Y}_2) &= \Sigma_{12}, \end{aligned}$$

where $\mathcal{C}_\mathrm{i}$ is the covariance matrix among p_i ordered independent standard normal variates.

Proof. If $\mathbf{Y}$ has the block-equicorrelated structure of $(\mathbf{Y}_1, \mathbf{Y}_2)$ then $\mathbf{Y} = \mathbf{m}+\mathbf{TU}$ for a suitable choice of $\mathbf{T}$ and $\mathbf{U} \sim \mathrm{N}(0, \mathbf{I})$, as shown in Example 2.2. In addition, the fact that the generating distribution is normal implies that $\mathrm{Cov}(\mathcal{U}_\mathrm{i})$ is stochastic [e.g., David (1981, p. 39)]. Applying Corollary 2.1, the result follows. □

To obtain a partial converse to Corollary 2.2 we need the following result:

Proposition 2.3. Let $\mathrm{Z}_1, \dots, \mathrm{Z}_\mathrm{p}$ be independent, identically distributed random variables with zero means and variance σ^2, $\mathrm{Z}_{(1)} \leq \ldots \leq \mathrm{Z}_{(\mathrm{p})}$ its ordered values and $\mathcal{Z} = (\mathrm{Z}_{(1)}, \dots, \mathrm{Z}_{(\mathrm{p})})$. Then $\mathrm{Cov}(\mathcal{Z})$ is doubly stochastic for all $\mathrm{p} = 2, 3, \dots$ if and only if the distribution of Z is normal.

Proof. The proof of doubly stochasticity for independent normal random variables is provided in David (1981, p. 39). Suppose, conversely, that $\mathrm{Cov}(\mathcal{Z})$ is doubly stochastic for all $\mathrm{p} = 2, 3, \dots$. Then

$$1 = \sum_{\mathrm{j}=1}^{\mathrm{p}} \mathrm{Cov}(\mathrm{Z}_{(\mathrm{i})}, \mathrm{Z}_{(\mathrm{j})}), \quad \mathrm{i} = 1, \dots, \mathrm{p}, \quad \mathrm{p} = 2, \dots.$$

But this is a characterization of the normal distribution (Govindarajulu 1966, Corollary 3.3.1) [see also Johnson and Kotz (1970, p. 53)]. □

Proposition 2.4. If $\mathbf{Y} = \mathbf{m} + \mathbf{TU}$ as in Proposition 2.2 with $\mathrm{U}_0 \sim \mathrm{N}(0, 1)$, $\mathrm{p}_1 = \mathrm{p}_2 = \mathrm{p}$, $\mathrm{a}\beta + \alpha\mathrm{b} \neq 0$ and $\Sigma_{12} = \Psi_{12}$ for all $\mathrm{p} \geq 2$ then $\mathbf{Y} \sim \mathrm{N}(\mathbf{m}, \mathbf{TT}^{'})$.

Proof. The conditions $\mathrm{p}_1 = \mathrm{p}_2 = \mathrm{p}$, $\mathrm{a}\beta + \alpha\mathrm{b} \neq 0$ and $\Sigma_{12} = \Psi_{12}$ for all $\mathrm{p} \geq 2$ imply that $\mathrm{Cov}(\mathcal{U}_\mathrm{i})$ is stochastic for all $\mathrm{p} \geq 2$. From Proposition 2.3 and using the fact that the components of $\mathbf{U}$ are jointly independent, we obtain that $\mathbf{U}_1$ and $\mathbf{U}_2$ are jointly normal and independent. Because, in addition, it is assumed that $\mathrm{U}_0 \sim \mathrm{N}(0, 1)$, the generating vector $\mathbf{U}$ is normal $\mathrm{N}(0, \mathbf{I})$ (of dimension $2\mathrm{p} + 1$). Therefore, $\mathbf{Y} \sim \mathrm{N}(\mathbf{m}, \mathbf{TT}^{'})$, concluding the proof. □

3. Correlation parameters of Ψ when $\phi = \gamma$.

From Corollary 2.2, the correlation $\eta_{\mathrm{ii,jk}}$ between $\mathrm{Y}_{\mathrm{i(j)}}$ and $\mathrm{Y}_{\mathrm{i(k)}}$ is

$$\eta_{\mathrm{ii,jk}} = \frac{\rho_\mathrm{i} + (1 - \rho_\mathrm{i})\mathrm{c}_{\mathrm{i,jk}}}{\sqrt{\rho_\mathrm{i} + (1 - \rho_\mathrm{i})\mathrm{c}_{\mathrm{i,jj}}}\sqrt{\rho_\mathrm{i} + (1 - \rho_\mathrm{i})\mathrm{c}_{\mathrm{i,kk}}}}, \tag{3.1}$$

where $\mathrm{c}_{\mathrm{i,jk}}$ is the jk$-$th entry of the covariance matrix $\mathcal{C}_\mathrm{i}$ among p_i independent standard normal variates, $\mathrm{i} = 1, 2$, $\mathrm{j} = 1, \dots, \mathrm{p}_1$, $\mathrm{k} = 1, \dots, \mathrm{p}_2$.

The correlation $\eta_{12,\mathrm{jk}}$ between $\mathrm{Y}_{1(\mathrm{j})}$ and $\mathrm{Y}_{2(\mathrm{k})}$ is

$$\eta_{12,\mathrm{jk}} = \frac{\gamma}{\sqrt{\rho_1 + (1 - \rho_1)\mathrm{c}_{1,\mathrm{jj}}}\sqrt{\rho_2 + (1 - \rho_2)\mathrm{c}_{2,\mathrm{kk}}}}. \tag{3.2}$$

The conditional covariance matrix $\Psi_{22.1} = \Psi_{22} - \Psi_{21}\Psi_{11}^{-1}\Psi_{12}$ is

$$\Psi_{22.1} = \Sigma_{22}\mathcal{C}_2 - \frac{\mathrm{p}_1\gamma^2}{\sigma_1^2(1 + (\mathrm{p}_1 - 1)\rho_1)}\mathbf{v}_{22}, \tag{3.3}$$

whereas the matrix $\Psi_{21}\Psi_{11}^{-1}$ of regression coefficients is

$$\Psi_{21}\Psi_{11}^{-1} = \frac{\gamma}{\sigma_1^2(1 + (\mathrm{p}_1 - 1)\rho_1)}\mathbf{v}_{21}.$$

4. Inferences on Σ

We assume that $p_1 = p_2 = p$. The matrix of variances and covariances among the 2p components of $(\mathbf{X}, \mathbf{Y})$ is determined by (1.2), where the range of the parameters is

$$[\gamma + (p-1)\phi]^2 < [1 + (p-1)\rho_1][1 + (p-1)\rho_2],$$
$$(\gamma - \phi)^2 < (1-\rho_1)(1-\rho_2),$$
$$\frac{-1}{p-1} < \rho_1 < 1, \ \frac{-1}{p-1} < \rho_2 < 1,$$

so as to guarantee that the covariance matrix Σ is positive definite. If the joint distribution of $(\mathbf{Y}, \mathbf{X})$ is multivariate normal with covariance matrix (1.2), then the maximum likelihood estimate $\hat{\Delta}$ of $\Delta = (\sigma_1^2, \sigma_2^2)$ is given by

$$\hat{\Delta} = (\frac{trS_{11}}{p}, \frac{trS_{22}}{p}),$$

where tr(S) indicates the trace of the corresponding sample covariance matrix S, whereas the maximum likelihood estimate $\hat{\Theta}$ of $\Theta = (\rho_1, \rho_2, \gamma, \phi)$ has components

$$\hat{\rho}_i = \frac{\bar{S}_{ii} - trS_i}{(p-1)trS_i}, \quad i = 1, 2, \tag{4.1}$$

$$\hat{\gamma} = \frac{tr(S_{12})}{\sqrt{trS_{22}}\sqrt{trS_{11}}}, \tag{4.2}$$

$$\hat{\phi} = \frac{\bar{S}_{12} - tr(S_{12})}{(p-1)\sqrt{trS_{22}}\sqrt{trS_{11}}}. \tag{4.3}$$

Here, and throughout the paper, $\bar{S}$ indicates the sum of all entries of the corresponding sample covariance matrix S.

When $\gamma = \phi$, the MLE $\hat{\gamma}$ of γ becomes

$$\hat{\gamma} = \frac{\bar{S}_{12}}{p\sqrt{trS_{22}}\sqrt{trS_{11}}},$$

a weighted combination of the unrestricted estimates $\hat{\gamma}$ and $\hat{\phi}$ in (4.3) with corresponding weights $p(p-1)$ and p.

5. Large-sample distributions.

Proposition 5.1. If the joint distribution of $(\mathbf{Y}, \mathbf{X})$ is multivariate normal with covariance matrix (1.2), then [i] the asymptotic joint distribution of $\sqrt{N}(\hat{\Delta} - \Delta)$ is bivariate normal with means zero, variances

$$\mathrm{var}_\infty(\hat{\sigma_i}^2) = \frac{2\sigma_i^4\left(1 + (p-1)\rho_i^2\right)}{p}, \quad i = 1, 2$$

and covariance

$$\mathrm{Cov}_\infty(\hat{\sigma}_1^2, \hat{\sigma}_2^2) = \frac{2\,\sigma_2^2\sigma_1^2\left(\gamma^2 + (p-1)\phi^2\right)}{p},$$

[ii] the asymptotic joint distribution of $\sqrt{\mathrm{N}}(\hat{\Theta}-\Theta)$ is 4-variate normal, with means zero, variances

$$\mathrm{var}_\infty(\hat{\rho}_\mathrm{i}) = \frac{2\,\left(1+(\mathrm{p}-1)\rho_\mathrm{i}\right)^2(1-\rho_\mathrm{i})^2}{\mathrm{p}(\mathrm{p}-1)}, \quad \mathrm{i}=1,2,$$

$$\begin{aligned}\mathrm{var}_\infty(\hat{\gamma}) =& [2+\gamma^2\rho_1^2\mathrm{p}-\gamma^2\rho_1^2+2\,\gamma^2\phi^2\mathrm{p}-2\,\gamma^2\phi^2+\gamma^2\rho_2^2\mathrm{p}\\ &-\gamma^2\rho_2^2-4\,\gamma\,\rho_1\,\mathrm{p}\phi+4\,\gamma\,\rho_1\,\phi-4\,\gamma^2+4\,\gamma\,\phi\,\rho_2-4\,\gamma\,\phi\,\mathrm{p}\rho_2\\ &+2\,\gamma^4+2\,\phi^2\mathrm{p}-2\,\phi^2+2\,\rho_1\,\mathrm{p}\rho_2-2\,\rho_1\,\rho_2]/(2\,\mathrm{p}),\end{aligned}$$

$$\begin{aligned}\mathrm{var}_\infty(\hat{\phi}) =& [2+2\,\gamma^2+12\,\phi^2+2\,\rho_2\,\mathrm{p}+2\,\rho_1\,\mathrm{p}-8\,\gamma\,\phi-4\,\rho_2-4\,\rho_1-12\,\phi^2\mathrm{p}+2\,\phi^4\mathrm{p}^2\\ &-2\,\phi^2\rho_2^2\mathrm{p}+\phi^2\rho_1^2\mathrm{p}^2+\phi^2\rho_2^2\mathrm{p}^2-4\,\phi^4\mathrm{p}+2\,\rho_1\,\mathrm{p}^2\rho_2-2\,\phi^2\rho_1^2\mathrm{p}\\ &+\phi^2\rho_2^2+\phi^2\rho_1^2+2\,\phi^4-4\,\gamma\,\phi\,\mathrm{p}\rho_2\\ &+12\,\rho_1\,\mathrm{p}\phi^2+12\,\phi^2\mathrm{p}\rho_2-4\,\phi^2\mathrm{p}^2\rho_2-4\,\rho_1\,\mathrm{p}^2\phi^2-4\,\gamma\,\rho_1\,\mathrm{p}\phi-2\,\gamma^2\phi^2\\ &+6\,\rho_1\,\rho_2-8\,\rho_1\,\phi^2-8\,\phi^2\rho_2+2\,\phi^2\mathrm{p}^2+2\,\gamma^2\phi^2\mathrm{p}+4\,\gamma\,\rho_1\,\phi\\ &+4\,\gamma\,\phi\,\rho_2-6\,\rho_1\,\mathrm{p}\rho_2+4\,\gamma\,\phi\,\mathrm{p}]/(2\,\mathrm{p}(\mathrm{p}-1)),\end{aligned}$$

and covariances

$$\begin{aligned}\mathrm{Cov}_\infty(\hat{\rho}_1,\hat{\rho}_2) =& [-8\,\gamma\,\phi-4\,\gamma\,\rho_1\,\mathrm{p}\phi+4\,\gamma\,\rho_1\,\phi+2\,\gamma^2+2\,\gamma^2\rho_1\,\mathrm{p}\rho_2\\ &-4\,\rho_1\,\mathrm{p}\phi^2\rho_2+4\,\gamma\,\phi\,\rho_2-2\,\gamma^2\rho_1\,\rho_2-4\,\gamma\,\phi\,\mathrm{p}\rho_2-6\,\phi^2\mathrm{p}\\ &+6\,\phi^2+6\,\rho_1\,\mathrm{p}\phi^2-4\,\rho_1\,\phi^2+6\,\phi^2\mathrm{p}\rho_2-4\,\phi^2\rho_2+2\,\phi^2\mathrm{p}^2+2\,\rho_1\,\phi^2\rho_2\\ &+2\,\rho_1\,\mathrm{p}^2\phi^2\rho_2-2\,\phi^2\mathrm{p}^2\rho_2-2\,\rho_1\,\mathrm{p}^2\phi^2+4\,\gamma\,\phi\,\mathrm{p}]/[\mathrm{p}(\mathrm{p}-1)],\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}_\infty(\hat{\rho}_\mathrm{i},\hat{\gamma}) =& [-4\,\rho_\mathrm{i}\,\phi-2\,\gamma^2\phi+\gamma^3\rho_\mathrm{i}-\gamma\,\rho_\mathrm{i}\,\phi^2+\gamma\,\rho_\mathrm{i}\,\mathrm{p}\phi^2\\ &-\gamma\,\phi^2\mathrm{p}-\gamma\,\rho_\mathrm{i}^3-\gamma\,\rho_\mathrm{i}-\gamma\,\rho_\mathrm{i}^2\mathrm{p}+2\,\gamma\,\rho_\mathrm{i}^2+\gamma\,\rho_\mathrm{i}^3\mathrm{p}+2\,\phi+2\,\rho_\mathrm{i}\,\mathrm{p}\phi\\ &+2\,\gamma\,\phi^2-2\,\phi\,\rho_\mathrm{i}^2\mathrm{p}+2\,\phi\,\rho_\mathrm{i}^2]/\mathrm{p}, \quad \mathrm{i}=1,2\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}_\infty(\hat{\rho}_\mathrm{i},\hat{\phi}) =& [2\,\gamma+9\,\rho_\mathrm{i}\,\phi+2\,\gamma\,\rho_\mathrm{i}^2-6\,\phi\,\rho_\mathrm{i}^2-4\,\phi-2\,\phi^3+\gamma^2\rho_\mathrm{i}\,\mathrm{p}\phi+\rho_\mathrm{i}\,\phi^3\\ &+\phi\,\rho_\mathrm{i}^3+3\,\phi^3\mathrm{p}+2\,\phi\,\mathrm{p}-\phi^3\mathrm{p}^2-2\,\gamma\,\phi^2\mathrm{p}+2\,\gamma\,\phi^2-9\,\rho_\mathrm{i}\,\mathrm{p}\phi+\phi\,\rho_\mathrm{i}^3\mathrm{p}^2\\ &-2\,\phi\,\rho_\mathrm{i}^3\mathrm{p}-2\,\gamma\,\rho_\mathrm{i}^2\mathrm{p}+2\,\rho_\mathrm{i}\,\mathrm{p}^2\phi+\rho_\mathrm{i}\,\mathrm{p}^2\phi^3-2\,\rho_\mathrm{i}\,\mathrm{p}\phi^3-\gamma^2\rho_\mathrm{i}\,\phi\\ &+9\,\phi\,\rho_\mathrm{i}^2\mathrm{p}-3\,\phi\,\rho_\mathrm{i}^2\mathrm{p}^2+2\,\gamma\,\rho_\mathrm{i}\,\mathrm{p}-4\,\gamma\,\rho_\mathrm{i}]/[\mathrm{p}(\mathrm{p}-1)], \quad \mathrm{i}=1,2,\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}_\infty(\hat{\gamma},\hat{\phi}) =& [-2\,\gamma^2\rho_2-2\,\gamma\,\phi+2\,\gamma^3\phi-2\,\gamma\,\rho_1\,\mathrm{p}\phi+4\,\gamma\,\rho_1\,\phi+4\,\gamma\,\phi\,\rho_2\\ &-2\,\gamma\,\phi\,\mathrm{p}\rho_2+2\,\rho_1+\phi\,\mathrm{p}\gamma\,\rho_2^2+\phi\,\mathrm{p}\gamma\,\rho_1^2-\phi\,\gamma\,\rho_1^2+2\,\phi^3\mathrm{p}\gamma\\ &-\phi\,\gamma\,\rho_2^2-2\,\gamma\,\phi^3+2\,\rho_2+2\,\phi^2\mathrm{p}-4\,\phi^2+2\,\rho_1\,\mathrm{p}\rho_2-4\,\rho_1\,\rho_2-2\,\rho_1\,\mathrm{p}\phi^2\\ &+2\,\rho_1\,\phi^2-2\,\phi^2\mathrm{p}\rho_2+2\,\phi^2\rho_2-2\,\gamma^2\rho_1]/(2\,\mathrm{p}).\end{aligned}$$

The proof is shown in Section 8.

5.1. **Special Cases.** Under the conditions of Proposition 5.1 we obtain, for example, for the bivariate case (p=2),

$$\mathrm{var}_\infty(\hat{\Delta}) = \begin{bmatrix} \sigma_1^4\left(\rho_1^2+1\right) & \sigma_2^2\sigma_1^2\left(\phi^2+\gamma^2\right) \\ \sigma_2^2\sigma_1^2\left(\phi^2+\gamma^2\right) & \sigma_2^4\left(\rho_2^2+1\right) \end{bmatrix},$$

$$\mathrm{var}_\infty(\hat{\rho}_1 \mid \rho_1 = 0) = \mathrm{var}_\infty(\hat{\rho}_2 \mid \rho_2 = 0) = \frac{2}{\mathrm{p}(\mathrm{p}-1)},$$

$$\mathrm{var}_\infty(\hat{\gamma} \mid \gamma = 0) = \frac{1+(\mathrm{p}-1)(\phi^2+\rho_1\rho_2)}{\mathrm{p}},$$

$$\mathrm{var}_\infty(\hat{\phi} \mid \phi = 0) = \frac{\mathrm{p}^2\rho_1\rho_2 - 3\mathrm{p}\rho_1\rho_2 + 3\rho_1\rho_2 - 2\rho_2 - 2\rho_1 + \gamma^2 + \mathrm{p}\rho_2 + \mathrm{p}\rho_1 + 1}{\mathrm{p}(\mathrm{p}-1)},$$

$$\mathrm{Cov}_\infty(\hat{\Theta} \mid \Theta = 0) = \mathrm{diag}(\frac{2}{\mathrm{p}(\mathrm{p}-1)}, \frac{2}{\mathrm{p}(\mathrm{p}-1)}, \frac{1}{\mathrm{p}}, \frac{1}{\mathrm{p}(\mathrm{p}-1)}).$$

The case $\rho_1 = \rho_2$ is important. However, Proposition 5.1 cannot be adapted easily. To see this, take the restricted case: $\mathrm{p} = 2, \gamma = \phi = 0$; then the MLE of the common correlation ρ is the solution of a quadratic equation [e.g. Viana (1994), Olkin and Pratt (1958),Olkin and Siotani (1964),Olkin (1967)].

From (3.1) and (3.2), note that when $\mathrm{p}_1 = \mathrm{p}_2 = 2$, the three distinct correlations of interest are

$$\begin{aligned} \eta_{11,12} &= \mathrm{Cor}(\mathrm{Y}_{1(1)}, \mathrm{Y}_{1(2)}), \\ \eta_{22,12} &= \mathrm{Cor}(\mathrm{Y}_{2(1)}, \mathrm{Y}_{2(2)}), \\ \eta_{12,\mathrm{jk}} &= \mathrm{Cor}(\mathrm{Y}_{1(\mathrm{j})}, \mathrm{Y}_{2(\mathrm{k})}), \quad \mathrm{j,k} = 1, 2. \end{aligned}$$

Let $\Lambda = (\eta^2_{11,12}, \eta^2_{22,12}, \eta^2_{12,\mathrm{jk}})$.

Proposition 5.2. If the distribution of $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2)$ is multivariate normal with permutation-symmetric bivariate means $\mathbf{m}_1$ and $\mathbf{m}_2$, and block-equicorrelated covariance structure (1.2) with $\phi = \gamma$, then the asymptotic joint distribution of $\sqrt{\mathrm{N}}(\hat{\Lambda} - \Lambda)$ is normal with means zero, variances

$$\mathrm{Var}_\infty(\hat{\eta}^2_{\mathrm{ii},12}) = \frac{(0.6817\rho_\mathrm{i} + 0.3183)^2\,(1+\rho_\mathrm{i})^2\,(0.2312\rho_\mathrm{i} + 0.4954)^2\,(1-\rho_\mathrm{i})^2}{(0.3183\rho_\mathrm{i} + 0.6817)^8},$$

$\mathrm{i} = 1, 2,$

$$\begin{aligned} \mathrm{Var}_\infty(\hat{\eta}^2_{12,\mathrm{jk}}) = {} & [(0.2028\gamma^4\rho_1\,\rho_2 - 0.899\gamma^2\rho_1\,{\rho_2}^2 + 3.230\rho_2 - 2.289\gamma^2 + 3.225\rho_1 \\ & - 3.173\gamma^2\rho_2 + 0.750{\rho_2}^2 + 0.4338\gamma^4\rho_2 - 0.716\gamma^2{\rho_2}^2 + 0.7575{\rho_1}^2 \\ & - 0.7171\gamma^2{\rho_1}^2 + 3.188{\rho_1}^2{\rho_2}^2 + 0.4342\gamma^4\rho_1 + 0.9292\gamma^4 - 3.933\gamma^2\rho_1\,\rho_2 \\ & + 6.472\rho_1\,\rho_2 + 3.928{\rho_1}^2\rho_2 - 3.177\gamma^2\rho_1 + 3.923\rho_1\,{\rho_2}^2 + 0.7575{\rho_1}^3\rho_2 \\ & - 0.8968\gamma^2{\rho_1}^2\rho_2 - 0.198\gamma^2{\rho_1}^2{\rho_2}^2 + 3.455 + 0.750\rho_1\,{\rho_2}^3 + 0.703{\rho_1}^3{\rho_2}^2 \\ & + 0.696{\rho_2}^3{\rho_1}^2 + 0.1688{\rho_1}^3{\rho_2}^3)\gamma^2] \\ & /[32\;(0.3183\rho_1 + 0.6817)^4\,(0.3183\rho_2 + 0.6817)^4], \end{aligned}$$

and covariances

$$\begin{aligned}\mathrm{Cov}_\infty(\hat\eta^2_{11,12},\hat\eta^2_{22,12}) = {}&[(0.492+0.237\rho_2)\,(1-\rho_2)\\ &(0.682\rho_2+0.3183)\,(0.682\rho_1+0.3183)\,(1-\rho_1)\\ &(0.492+0.237\rho_1)\,\gamma^2]\\ &/[2\ (0.318\rho_2+0.6817)^4\,(0.318\rho_1+0.6817)^4],\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}_\infty(\hat\eta^2_{\mathrm{ii},12},\hat\eta^2_{12,\mathrm{jk}}) = {}&[(-0.03\rho_1\,{\rho_2}^2-0.06{\rho_2}^2+1.43-0.318\gamma^2\rho_2-0.6817\gamma^2\\ &+0.66\rho_1+1.36\rho_2+0.64\rho_1\,\rho_2)\,(0.682\rho_2+0.3183)\\ &(1-\rho_2)\,(0.492+0.237\rho_2)\,\gamma^2]\\ &/[8\ (0.318\rho_1+0.6817)^2\,(0.318\rho_2+0.6817)^6].\end{aligned}$$

In particular, under the conditions of Proposition 5.2, when $\rho_1 = \rho_2 = 0$ we obtain

$$\begin{aligned}\mathrm{Var}_\infty(\hat\eta^2_{\mathrm{ii},12}) &= 0.5330,\ \ \mathrm{i}=1,2\\ \mathrm{Var}_\infty(\hat\eta^2_{12,\mathrm{jk}}) &= 0.6701\left(0.9288\gamma^4+3.455-2.291\gamma^2\right)\gamma^2\\ \mathrm{Cov}_\infty(\hat\eta^2_{11,12},\hat\eta^2_{22,12}) &= 0.2666\gamma^2\\ \mathrm{Cov}_\infty(\hat\eta^2_{\mathrm{ii},12},\hat\eta^2_{12,\mathrm{jk}}) &= 0.4227\left(1.423-0.6813\gamma^2\right)\gamma^2,\ \ \mathrm{i}=1,2.\end{aligned}$$

6. Intra-ocular Pressure Data

In the study described by Sonty, Sonty and Viana (1996), intra-ocular pressure (IOP) measurements at pre-treatment ($\mathbf{Y}_1$) and post treatment ($\mathbf{Y}_2$) conditions were obtained from fellow glaucomatous eyes of N = 15 subjects on topical beta blocker therapy. We want to estimate the covariance structure between pre and post-treatment ordered IOP. This is important to assess and linearly predict the best (worst) post-treatment IOP from pre-treatment ocular pressure measurements. The observed covariance matrices for IOP between fellow eyes are:

$$\mathrm{S}_{11} = \begin{bmatrix} 12.410 & 7.019\\ 7.019 & 12.924\end{bmatrix},\quad \mathrm{S}_{22} = \begin{bmatrix} 17.029 & 15.371\\ 15.371 & 17.352\end{bmatrix},$$

whereas the sample cross-covariance matrix is

$$\mathrm{S}_{12} = \begin{bmatrix} 11.671 & 9.348\\ 8.200 & 10.076\end{bmatrix}.$$

From Section 4, the estimated variances are

$$\hat\sigma_1^2 = \frac{\mathrm{trS}_{11}}{\mathrm{p}} = 12.667,\quad \hat\sigma_2^2 = \frac{\mathrm{trS}_{22}}{\mathrm{p}} = 17.190,$$

whereas the estimated correlations are

$$\hat\rho_1 = \frac{\bar{\mathrm{S}}_{11}-\mathrm{trS}_{11}}{(\mathrm{p}-1)\mathrm{trS}_{11}} = 0.553,\ \ \hat\rho_2 = \frac{\bar{\mathrm{S}}_{22}-\mathrm{trS}_{22}}{(\mathrm{p}-1)\mathrm{trS}_{22}} = 0.894,$$

$$\hat\gamma = \frac{\mathrm{tr(S_{12})}}{\sqrt{\mathrm{trS}_{22}}\sqrt{\mathrm{trS}_{11}}} = 0.736,\ \ \hat\phi = \frac{\bar{\mathrm{S}}_{12}-\mathrm{tr(S_{21})}}{(\mathrm{p}-1)\sqrt{\mathrm{trS}_{22}}\sqrt{\mathrm{trS}_{11}}} = 0.594.$$

The corresponding large-sample estimated covariance matrices are

$$\mathrm{Cov}_\infty(\hat{\Delta}) = \begin{bmatrix} 209.503 & 194.764 \\ 194.764 & 531.624 \end{bmatrix},$$

$$\mathrm{Cov}_\infty(\hat{\Theta}) = \begin{bmatrix} 0.481 & 0.071 & 0.131 & 0.284 \\ 0.071 & 0.040 & 0.025 & 0.072 \\ 0.131 & 0.025 & 0.141 & 0.174 \\ 0.284 & 0.072 & 0.174 & 0.287 \end{bmatrix}.$$

The large-sample approximate 95% confidence intervals for γ and ϕ are, respectively, $(0.543, 0.929)$ and $(0.324, 0.864)$, thus suggesting a common cross-correlation γ. Under the restricted model $(\gamma = \phi)$ we have

$$\hat{\gamma} = \frac{\bar{\mathrm{S}}_{12}}{\mathrm{p}\sqrt{\mathrm{trS}_{22}}\sqrt{\mathrm{trS}_{11}}} = 0.665.$$

6.1. **Inferences on Ψ under $\phi = \gamma$.** The values of $\mathcal{C}_\mathrm{i}$ are tabulated, e.g., Sarhan and Greenberg (1962); Owen (1962). When $\mathrm{p}_1 = \mathrm{p}_2 = 2$ note that $\mathrm{c}_{\mathrm{i,jj}} = 0.6817$, $\mathrm{c}_{\mathrm{i},12} = 1 - \mathrm{c}_{\mathrm{i,jj}} = 0.3183$, $\mathrm{i,j} = 1, 2$. From equation (3.1), the estimated correlations $\eta_{\mathrm{ii,jk}}$ between $\mathrm{Y}_{\mathrm{i(j)}}$ and $\mathrm{Y}_{\mathrm{i(k)}}$ are

$$\hat{\eta}_{11,12} = \frac{\hat{\rho}_1 + (1-\hat{\rho}_1)0.3183}{\hat{\rho}_1 + (1-\hat{\rho}_1)0.6817} = 0.8106,$$
$$\hat{\eta}_{22,12} = \frac{\hat{\rho}_2 + (1-\hat{\rho}_2)0.3183}{\hat{\rho}_2 + (1-\hat{\rho}_2)0.6817} = 0.9463;$$

From equation (3.2), the estimated correlation $\hat{\eta}_{12,\mathrm{jk}}$ between $\mathrm{Y}_{1(\mathrm{j})}$ and $\mathrm{Y}_{2(\mathrm{k})}$ is

$$\hat{\eta}_{12,\mathrm{jk}} = \frac{\hat{\gamma}}{\sqrt{\hat{\rho}_1 + (1-\hat{\rho}_1)0.6817}\sqrt{\hat{\rho}_2 + (1-\hat{\rho}_2)0.6817}} = 0.7347.$$

From (3.3), the estimated conditional covariance matrix $\hat{\Psi}_{22.1}$ is

$$\hat{\Psi}_{22.1} = \begin{bmatrix} 16.562 & 15.858 \\ 15.858 & 16.562 \end{bmatrix}.$$

From Proposition 5.2 the asymptotic joint distribution of the squared-correlations $\hat{\Lambda} = (\hat{\eta}^2_{11,12}, \hat{\eta}^2_{22,12}, \hat{\eta}^2_{12,\mathrm{jk}})$ is normal with mean Λ and covariance matrix

$$\frac{1}{15}\begin{bmatrix} 0.3090 & 0.0062 & 0.0771 \\ 0.0062 & 0.0225 & 0.0180 \\ 0.0771 & 0.0180 & 0.3571 \end{bmatrix}.$$

The resulting large-sample approximate 95% confidence intervals for $\eta_{11,12}$, $\eta_{22,12}$ and $\eta_{12,\mathrm{jk}}$ are, respectively, $(0.608, 0.971)$, $(0.904, 0.986)$ and $(0.480, 0.920)$. The results suggest a positive association between pre and post ordered IOP measurements, estimated as 0.7374. The estimated correlation between pre-treatment extreme IOP measurements is 0.81, whereas the estimated correlation between post-treatment extreme IOPs is 0.94. Note that

$$\eta_{\mathrm{ii},12} = 0.467, \quad \text{when } \rho_\mathrm{i} = 0, \ \ \mathrm{i} = 1, 2,$$

which is the well known positive dependence between extreme fellow observations [e.g., Tong (1990, p. 130)]

7. Conclusions

We have described the covariance structure between the order statistics of p_1 permutation-symmetric normal variates $\mathbf{Y}_1$ and the order statistics of p_2 permutation-symmetric normal variates $\mathbf{Y}_2$, when $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are jointly normally distributed with constant cross-covariance Σ_{12} structure. The central results shows that the covariance structure of ordered fellow observations $\mathbf{Y}_{\mathrm{i}}$ (e.g., fellow eyes) has the form $\Sigma_{\mathrm{ii}}\mathcal{C}$ where Σ_{ii} is the permutation symmetric covariance structure of fellow observations and $\mathcal{C}$ is the doubly symmetric stochastic covariance structure of ordered independent permutation symmetric normal random variates. In addition, the cross-covariance structure between ordered $\mathbf{Y}_1$ and ordered $\mathbf{Y}_2$ remains equal to the original cross-covariance structure Σ_{12}.

8. Derivations

To prove Proposition 5.1 we need to express the distribution of the sample covariance matrix S in a suitable canonical form. We start with the assumption that

$$\mathrm{nS} = \mathrm{n}\begin{bmatrix} \mathrm{S}_{11} & \mathrm{S}_{12} \\ \mathrm{S}_{21} & \mathrm{S}_{22} \end{bmatrix}$$

has a Wishart distribution $\mathrm{W}_{2\mathrm{p}}(\Sigma, \mathrm{n})$, where

$$\Sigma = \mathrm{D}\begin{bmatrix} \rho_1\mathbf{ee}' + (1-\rho_1)\mathbf{I} & \phi\mathbf{ee}' + (\gamma-\phi)\mathbf{I} \\ \phi\mathbf{ee}' + (\gamma-\phi)\mathbf{I} & \rho_2\mathbf{ee}' + (1-\rho_2)\mathbf{I} \end{bmatrix}\mathrm{D},$$

and $\mathrm{D} = \mathrm{diag}(\sigma_1, \ldots, \sigma_1, \sigma_2, \ldots, \sigma_2)$. Let

$$\Gamma = \begin{bmatrix} \mathrm{Q} & 0 \\ 0 & \mathrm{Q} \end{bmatrix},$$

where Q is any $\mathrm{p} \times \mathrm{p}$ orthonormal real matrix with first row constant and equal to $1/\sqrt{\mathrm{p}}$. Then

$$\Gamma\Sigma\Gamma' = \mathrm{D}\begin{bmatrix} \mathrm{M}_{11} & \mathrm{M}_{12} \\ \mathrm{M}_{21} & \mathrm{M}_{22} \end{bmatrix}\mathrm{D},$$

where

$$\mathrm{M}_{\mathrm{ii}} = \mathrm{diag}(1 + (\mathrm{p}-1)\rho_{\mathrm{i}}, 1-\rho_{\mathrm{i}}, \ldots, 1-\rho_{\mathrm{i}}),$$
$$\mathrm{M}_{12} = \mathrm{diag}(\gamma + (\mathrm{p}-1)\phi, \gamma-\phi, \ldots, \gamma-\phi).$$

After transposition, we obtain

$$\Sigma_0 = \Gamma_0\Sigma\Gamma_0' = \mathrm{D}_0\ \mathrm{diag}(\mathrm{E}, \mathrm{F}, \ldots, \mathrm{F})\ \mathrm{D}_0,$$

where

$$\mathrm{E} = \begin{bmatrix} 1 + (\mathrm{p}-1)\rho_1 & \gamma + (\mathrm{p}-1)\phi \\ \gamma + (\mathrm{p}-1)\phi & 1 + (\mathrm{p}-1)\rho_2 \end{bmatrix}, \quad \mathrm{F} = \begin{bmatrix} 1-\rho_1 & \gamma-\phi \\ \gamma-\phi & 1-\rho_2 \end{bmatrix},$$

and $\mathrm{D}_0 = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_1, \sigma_2)$. Accordingly, $\mathrm{S}^* = \Gamma_0\mathrm{S}\Gamma_0'$ has a Wishart distribution $\mathrm{W}_{2\mathrm{p}}(\Sigma_0/\mathrm{n}, \mathrm{n})$, which can be decomposed into p independent Wishart components of dimension 2, corresponding to the non-zero block diagonals of Σ_0. Specifically, we obtain,

$$\mathrm{E}^* = \begin{bmatrix} \mathrm{S}^*_{11} & \mathrm{S}^*_{12} \\ \mathrm{S}^*_{21} & \mathrm{S}^*_{22} \end{bmatrix} \sim \mathrm{W}_2(\mathrm{E}/\mathrm{n}, \mathrm{n}),$$

$$\mathrm{F}_j^* = \begin{bmatrix} \mathrm{S}^*_{2j-1,2j-1} & \mathrm{S}^*_{2j-1,2j} \\ \mathrm{S}^*_{2j,2j-1} & \mathrm{S}^*_{2j,2j} \end{bmatrix} \sim \mathrm{W}_2(\mathrm{F}/n, n), \quad j = 2, \dots, p,$$

with $\mathrm{M}_1, \mathrm{M}_2, \dots, \mathrm{M}_p$ jointly independent. The canonical form of interest is then $n\mathrm{E}^* \sim \mathrm{W}_2(\mathrm{E}, n)$, independent of

$$n\mathrm{F}^* = n \sum_{j=2}^{p} \mathrm{F}_j^* \sim \mathrm{W}_2(\mathrm{F}, (p-1)n).$$

Let

$$\beta = \begin{bmatrix} \mathrm{E} \\ \cdots\cdots \\ (p-1)\mathrm{F} \end{bmatrix} = \begin{bmatrix} \sigma_1^2\,(1+(p-1)\,\rho_1) \\ \sigma_1\,\sigma_1\;(\gamma+(p-1)\,\phi) \\ \sigma_2^2\,(1+(p-1)\,\rho_2) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (p-1)\,\sigma_1^2\,(1-\rho_1) \\ (p-1)\,\sigma_1\,\sigma_2\;(\gamma-\phi) \\ (p-1)\,\sigma_2^2\,(1-\rho_2) \end{bmatrix}.$$

It then follows that $\Delta = \mathbf{f}(\beta)$ is determined by

$$\sigma_1^2 = \frac{\beta_1+\beta_4}{p}, \quad \sigma_2^2 = \frac{\beta_3+\beta_6}{p},$$

whereas $\Theta = \mathbf{g}(\beta)$ is determined by

$$\rho_1 = \frac{(p-1)\beta_1-\beta_4}{(p-1)(\beta_1+\beta_4)}, \quad \rho_2 = \frac{(p-1)\beta_3-\beta_6}{(p-1)(\beta_3+\beta_6)},$$

$$\gamma = \frac{\beta_2+\beta_5}{\sqrt{\beta_1+\beta_4}\sqrt{\beta_3+\beta_6}}, \quad \phi = \frac{(p-1)\beta_2-\beta_5}{(p-1)\sqrt{\beta_1+\beta_4}\sqrt{\beta_3+\beta_6}}.$$

In particular, when $\gamma = \phi$,

$$\gamma = \frac{\beta_2}{\sqrt{\beta_1+\beta_4}\sqrt{\beta_3+\beta_6}}.$$

Maximum Likelihood Estimates. Direct computation shows that

$$\mathrm{tr}(\mathrm{S}_{11}) = \sum_{j=1,3,2p-1} \mathrm{S}^*_{j,j},$$

$$\mathrm{tr}(\mathrm{S}_{22}) = \sum_{j=2,4,2p} \mathrm{S}^*_{j,j},$$

$$\mathrm{tr}(\mathrm{S}_{12}) = \sum_{j=1,3,2p-1} \mathrm{S}^*_{j,j+1},$$

$$\bar{\mathrm{S}}_{11} - \mathrm{tr}(\mathrm{S}_1) = (p-1)\mathrm{S}^*_{11} - \sum_{j=3,5,2p-1} \mathrm{S}^*_{j,j},$$

$$\bar{\mathrm{S}}_{22} - \mathrm{tr}(\mathrm{S}_2) = (p-1)\mathrm{S}^*_{22} - \sum_{j=4,6,2p} \mathrm{S}^*_{j,j},$$

$$\bar{\mathrm{S}}_{12} - \mathrm{tr}(\mathrm{S}_{12}) = (p-1)\mathrm{S}^*_{12} - \sum_{j=3,5,2p-1} \mathrm{S}^*_{j,j+1},$$

so that

$$\hat{\mathrm{E}} = \frac{1}{\mathrm{p}} \begin{bmatrix} \bar{\mathrm{S}}_{11} & \bar{\mathrm{S}}_{12} \\ \bar{\mathrm{S}}_{21} & \bar{\mathrm{S}}_{22} \end{bmatrix},$$

$$\mathrm{p}\hat{\mathrm{F}} = \begin{bmatrix} \mathrm{ptr}(\mathrm{S}_{11}) - \bar{\mathrm{S}}_{11} & \mathrm{ptr}(\mathrm{S}_{12}) - \bar{\mathrm{S}}_{12} \\ \mathrm{ptr}(\mathrm{S}_{21}) - \bar{\mathrm{S}}_{21} & \mathrm{ptr}(\mathrm{S}_{22}) - \bar{\mathrm{S}}_{22} \end{bmatrix},$$

thus showing that the estimate $\hat{\beta} = (\hat{\mathrm{E}}, \hat{\mathrm{F}})$ of β does not depend of the particular choice of Γ_0. The estimates indicated on Proposition 5.1 follow by expressing the corresponding MLEs $\hat{\Delta} = \mathbf{f}(\hat{\beta}), \quad \hat{\Theta} = \mathbf{g}(\hat{\beta})$ accordingly.

Large-sample distributions. Because

$$\sqrt{\mathrm{n}}(\mathrm{E}^* - \mathrm{E}) \ \stackrel{\mathcal{L}}{\to}\ \mathrm{N}_2(0, \mathrm{T}_1),$$

independent of

$$\sqrt{\mathrm{n}}(\mathrm{F}^* - (\mathrm{p}-1)\mathrm{F}) \ \stackrel{\mathcal{L}}{\to}\ \mathrm{N}_2(0, (\mathrm{p}-1)\mathrm{T}_2),$$

where T_1 is the covariance matrix of E^* and T_2 is the common covariance matrix of F_j^*, $\mathrm{j} = 2\ldots,\mathrm{p}$. It then follows that

$$\sqrt{\mathrm{n}}(\begin{bmatrix} \mathrm{E}^* \\ \mathrm{F}^* \end{bmatrix} - \beta) \ \stackrel{\mathcal{L}}{\to}\ \mathrm{N}_4(0, \mathrm{T}),$$

where T is the block matrix with diagonal $\mathrm{T}_1, (\mathrm{p}-1)\mathrm{T}_2$. The limiting joint distribution of $\hat{\rho_2} = \mathbf{f}(\mathrm{M})$ and $\hat{\Theta} = \mathbf{g}(\mathrm{M})$ follows from the standard delta method.

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