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THE JOINT COVARIANCE STRUCTURE OF ORDERED SYMMETRICALLY DEPENDENT OBSERVATIONS AND THEIR CONCOMITANTS OF ORDER STATISTICS

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Abstract. We considered the ordered components, $\mathcal{Y}$, of a multivariate random variable, $\mathbf{Y}$, with covariance matrix $\mathbf{\Sigma}_{11}$ and the vector, $\mathcal{Z}$, of concomitantly or induced ordered components of a secondary random vector, $\mathbf{Z}$, with covariance matrix $\mathbf{\Sigma}_{22}$. Assuming that $\mathbf{\Sigma}_{11}$, $\mathbf{\Sigma}_{22}$ and the covariance structure between $\mathbf{Y}$ and $\mathbf{Z}$ are permutation-symmetric, the joint covariance structure for $\mathcal{Y}$ and $\mathcal{Z}$ is obtained. The case in which the joint probability distribution of $(\mathbf{Y}, \mathbf{Z})$ is multivariate normal leads to an explicit formulation of the covariances of interest.

1. Introduction

Let $\mathbf{Y}$ and $\mathbf{Z}$ indicate jointly observed random vectors in R^{p} with covariance structure defined by

$$\mathbf{\Sigma}_{11} = \mathrm{Cov}(\mathbf{Y}) = \sigma_1^2(\gamma_{11}\mathbf{e}\mathbf{e}^{'} + (1-\gamma_{11})\mathbf{I}),$$

$$\mathbf{\Sigma}_{22} = \mathrm{Cov}(\mathbf{Z}) = \sigma_2^2(\gamma_{22}\mathbf{e}\mathbf{e}^{'} + (1-\gamma_{22})\mathbf{I}),$$

$$\mathbf{\Sigma}_{12} = \mathrm{Cov}(\mathbf{Y}, \mathbf{Z}) = \sigma_1\sigma_2(\gamma_{12}\mathbf{e}\mathbf{e}^{'} + (\lambda_{12}-\gamma_{12})\mathbf{I}),$$

where $\mathbf{e}^{'} = (1, \ldots, 1)$ has p components and $\mathbf{I}$ is the identity matrix of corresponding dimension. When the block components $\mathbf{\Sigma}_{11}, \mathbf{\Sigma}_{12}, \mathbf{\Sigma}_{22}$ of $\mathbf{\Sigma} = \mathrm{Cov}[(\mathbf{Y}, \mathbf{Z})]$ satisfy these symmetry conditions, we say that $\mathbf{\Sigma}$ has a block-permutation symmetry (**BPS**). Similarly, we say that the vector of means $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \boldsymbol{\mu}_2)$ satisfies the **BPS** condition when $\boldsymbol{\mu}_1 = \mathrm{m}_1\mathbf{e}$ and $\boldsymbol{\mu}_2 = \mathrm{m}_2\mathbf{e}$, for scalars $\mathrm{m}_1, \mathrm{m}_2$. We remark that $\mathrm{Cov}[(\mathbf{Y}, \mathbf{Z})]$ indicates the block-covariance matrix $\mathbf{\Sigma}$, whereas $\mathrm{Cov}(\mathbf{Y}, \mathbf{Z})$ denotes the cross-covariance block $\mathbf{\Sigma}_{12}$ of $\mathbf{\Sigma}$.

We indicate by $\mathcal{Y}^{'} = (\mathrm{Y}_{(1)}, \mathrm{Y}_{(2)}, \ldots, \mathrm{Y}_{(\mathrm{p})})$ the ordered version of $\mathbf{Y}$, understanding that $\mathrm{Y}_{(1)} \leq \ldots \leq \mathrm{Y}_{(\mathrm{p})}$. Correspondingly, $\mathcal{Z}$ indicates the vector of induced or concomitant order statistics. The component $\mathrm{Z}_{[\mathrm{i}]}$ of $\mathcal{Z}$ is defined by $\mathrm{Z}_{[\mathrm{i}]} = \mathrm{Z}_{\mathrm{j}}$ whenever $\mathrm{Y}_{(\mathrm{i})} = \mathrm{Y}_{\mathrm{j}}$. Then, $\mathcal{Z}$ is well-defined, provided

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that the underlying probability model of $\mathbf{Y}$ assigns probability zero to ties. If F indicates the joint probability distribution of $(\mathbf{Y},\mathbf{Z})$, and $\mathrm{F}(\mathrm{g}\mathbf{y},\mathrm{g}\mathbf{z}) = \mathrm{F}(\mathbf{y},\mathbf{z})$ for all $\mathrm{p}\times\mathrm{p}$ permutation matrices g, we say that F has the **BPS** property. In this case, it follows that, for all g, $\mathrm{g}\boldsymbol{\mu}_{\mathrm{i}} = \boldsymbol{\mu}_{\mathrm{i}},\ \mathrm{g}\boldsymbol{\Sigma}_{\mathrm{ij}}\mathrm{g}' = \boldsymbol{\Sigma}_{\mathrm{ij}},\ \mathrm{i},\mathrm{j}=1,2$. This implies that both $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ satisfy the **BPS** conditions.

In this report, we describe the joint covariance structure of $(\mathcal{Y},\mathcal{Z})$ when the underlying joint distribution of $(\mathbf{Y},\mathbf{Z})$ has the **BPS** property.

The standard formulation of concomitants is usually defined for N independent (e.g., a sample) observations from the underlying probability model of a bivariate vector with covariance structure Ψ. In that formulation, $\mathcal{Y}$ indicates the ordered sample and $\mathcal{Z}$ the concomitants. The resulting covariance structure $\boldsymbol{\Sigma}$ may be expressed in block form with $\boldsymbol{\Sigma}_{12} = \gamma\sigma_1\sigma_2\mathbf{I}$ and $\boldsymbol{\Sigma}_{\mathrm{ii}} = \sigma_{\mathrm{i}}^2\mathbf{I},\ \mathrm{i}=1,2$, where σ_1^2,σ_2^2 and γ are the parameters defining the bivariate structure in Ψ. The standard theory has its roots in David and Galambos (1974) and Bhattacharya (1976, 1984). A recent and comprehensive account of developments in the area is David and Nagaraja (1998). Earlier developments of the present formulation, with emphasis on the covariance structure ordered symmetrically dependent observations, may be seen in Olkin and Viana (1995), David (1996), Viana and Olkin (1997) and Viana (1998). In the present we consider the joint covariance structure of concomitants and orderd statistics of symmetrically dependent observations.

2. The joint covariance structure of $(\mathcal{Y},\mathcal{Z})$

Throughout this section, we assume that the joint probability distribution of $(\mathbf{Y},\mathbf{Z})$, F, has the **BPS** property and that $\mathcal{Z}$, the induced order statistics, is well-defined. We start with the following remark: *If* $\mathbf{Y}$ *and* $\mathbf{Z}$ *are jointly independent, then* $\mathcal{Y}$ *and* $\mathcal{Z}$ *are jointly independent and* $\mathbf{Z}\equiv\mathcal{Z}$ *in distribution.* In fact, if F has the **BPS** property, then, marginally, $\mathrm{F}_{\mathbf{y}}$ and $\mathrm{F}_{\mathbf{z}}$ are permutation symmetric and the distribution of $\mathcal{Z}$ can be represented by the distribution of $\mathbf{UZ}$ where $\mathbf{U}$ is a random permutation matrix representing the permutation of $\{1,2,\ldots,\mathrm{p}\}$ generated by ranking the components of $\mathbf{Y}$. Moreover, $\mathbf{U}$ is uniform in its class, depends on $\mathbf{Y}$ alone and hence is independent of $\mathcal{Z}$. This implies that $\mathcal{Z}\equiv\mathbf{UZ}\equiv\mathbf{Z}$ in distribution. It also implies that $\mathcal{Y}$ and $\mathcal{Z}$ are jointly independent. The representation $\mathbf{UZ}$ is discussed in more detail in Section 3.

Proposition 2.1. Suppose that $\mathbf{Y}$ and $\mathbf{Z}$ are related by $\mathbf{Z} = \mathbf{TY} + \mathbf{V}$, where $\mathbf{V}$ and $\mathbf{Y}$ are jointly independent and $\mathbf{T}$ is a permutation symmetric constant matrix (and hence $\mathbf{T} = \mathbf{T}'$). Then

$$\mathrm{Cov}[(\mathcal{Y}, \mathcal{Z})] = \begin{bmatrix} \mathbf{\Gamma} & \mathbf{\Gamma T} \\ \\ \mathbf{T\Gamma} & \mathbf{\Sigma}_{22} + \mathbf{T}(\mathbf{\Gamma} - \mathbf{\Sigma}_{11})\mathbf{T} \end{bmatrix}, \tag{1}$$

where $\mathbf{\Gamma} = \mathrm{Cov}(\mathcal{Y})$.

Proof. Note that $\mathbf{T} = \mathbf{T}'$ and write

$$\begin{bmatrix} \mathbf{Y} \\ \mathbf{Z} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{Y} \\ \mathbf{V} \end{bmatrix}, \tag{2}$$

with

$$\mathbf{A} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{T} & \mathbf{I} \end{bmatrix}. \tag{3}$$

Then, using the uniform representation described in the above remark,

$$\mathcal{Z} \equiv \mathbf{UZ} = \mathbf{U}(\mathbf{TY} + \mathbf{V}) = \mathbf{TUY} + \mathbf{UV} = \mathbf{T}\mathcal{Y} + \mathcal{V},$$

where $\mathcal{V}$ is the concomitant ordered version of $\mathbf{V}$ (note that $\mathbf{UT} = \mathbf{TU}$, because $\mathbf{T}$ is permutation symmetric and hence commutes with every permutation matrix of corresponding dimension). Consequently,

$$\begin{bmatrix} \mathcal{Y} \\ \mathcal{Z} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathcal{Y} \\ \mathcal{V} \end{bmatrix}. \tag{4}$$

Moreover, because $\mathbf{Y}$ and $\mathbf{V}$ are jointly independent, $\mathrm{Cov}(\mathbf{V}) = \mathrm{Cov}(\mathcal{V}) = \mathbf{\Sigma}_{22} - \mathbf{T\Sigma}_{11}\mathbf{T}$ so that

$$\mathrm{Cov}[(\mathcal{Y}, \mathcal{Z})] = \mathbf{A} \begin{bmatrix} \mathbf{\Gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_{22} - \mathbf{T\Sigma}_{11}\mathbf{T} \end{bmatrix} \mathbf{A}', \tag{5}$$

from which the result follows. □

Under the condition of Proposition 2.1, note that

$$\mathrm{E}(\mathcal{Z}) = \boldsymbol{\mu}_2\mathbf{e} + \mathbf{T}(\mathrm{E}(\mathcal{Y}) - \boldsymbol{\mu}_1), \tag{6}$$

which follows directly from $\mathcal{Z} = \mathbf{T}\mathcal{Y} + \mathcal{V}$ and $\mathrm{E}(\mathbf{V}) = \boldsymbol{\mu}_2 - \mathbf{T}\boldsymbol{\mu}_1$.

Corollary 2.1. If the joint distribution of $(\mathbf{Y}, \mathbf{Z})$ is multivariate normal satisfying the **BPS** condition, then

$$\mathrm{Cov}[(\mathcal{Y}, \mathcal{Z})] = \begin{bmatrix} \mathbf{\Sigma}_{11}\mathcal{C} & \mathbf{\Sigma}_{12}\mathcal{C} \\ \mathbf{\Sigma}_{21}\mathcal{C} & \mathbf{\Sigma}_{22} + \frac{\sigma_2^2(\lambda_{12}-\gamma_{12})^2}{(1-\gamma_{11})}(\mathcal{C} - \mathbf{I}) \end{bmatrix}, \quad (7)$$

where $\mathcal{C}$ is the covariance matrix of p ordered independent standard normal variates.

Proof. In Proposition 2.1, let $\mathbf{T} = \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}$ so that $\mathbf{T}$ is permutation symmetric, $\mathrm{Cov}(\mathbf{Z} - \mathbf{TY}, \mathbf{Y}) = 0$ and hence, $\mathbf{V} = \mathbf{Z} - \mathbf{TY}$ and $\mathbf{Y}$ are jointly independent. In addition, from Olkin and Viana (1995), we have $\mathrm{Cov}(\mathcal{Y}) = \mathbf{\Sigma}_{11}\mathcal{C}$. Direct evaluation of expression (1) then leads to the proposed result. □

It also follows, from Corollary 2.1, that

$$\mathrm{E}(\mathcal{Z}) = \boldsymbol{\mu}_2 + \frac{\sigma_2(\lambda_{12} - \gamma_{12})}{\sqrt{1 - \gamma_{11}}}\mathbf{c}, \quad (8)$$

where $\mathbf{c}$ is the mean vector of p ordered independent standard normal variates (use the fact, e.g., Owen and Steck (1962), that $\mathrm{E}(\mathcal{Y}) = \boldsymbol{\mu}_1 + \sigma_1\sqrt{1 - \gamma_{11}}\mathbf{c}$).

3. Comments

In Section 1, it is assumed that $\mathbf{\Sigma}$ is positive definite. A sufficient condition for $\mathbf{\Sigma} > 0$ is given by

$$[\lambda_{12} + (\mathrm{p}-1)\gamma_{12}]^2 < [1 + (\mathrm{p}-1)\gamma_{11}][1 + (\mathrm{p}-1)\gamma_{22}],$$
$$(\lambda_{12} - \gamma_{12})^2 < (1 - \gamma_{11})(1 - \gamma_{22}),$$
$$\frac{-1}{\mathrm{p}-1} < \gamma_{\mathrm{ii}} < 1 \ , \mathrm{i} = 1, 2 \ .$$

In Section 2, we represented the concomitants by $\mathbf{UZ}$, where $\mathbf{U}$ is a random permutation matrix. More specifically, let i' indicate the rank of the i-th component $\mathrm{Y_i}$ of $\mathbf{Y} \in \mathcal{R}^{\mathrm{p}}$ when these components are ordered from the smallest to the largest value, and assume that the components of $\mathbf{Y}$ are distinct. Let π indicate the permutation (1-1 bijection function) $\mathrm{i}' \to \mathrm{i}$. Given the standard basis $\{\mathrm{e}_1, \ldots, \mathrm{e}_{\mathrm{p}}\}$ of $\mathcal{R}^{\mathrm{p}}$, define the $\mathrm{p} \times \mathrm{p}$ matrix $\rho(\pi)$ by

$$\mathrm{e}_{\pi(\mathrm{j})} = \sum_{\mathrm{i}} \rho(\pi)_{\mathrm{ij}}\mathrm{e}_{\mathrm{i}}, \ \mathrm{j} = 1, \ldots, \mathrm{p}.$$

The matrix $\mathbf{U} = \rho^{-1}(\pi)$ is called the *order representation* of (the ranks of) $\mathbf{Y}$, is a random permutation matrix, and $\mathbf{UY}$ generates the ordered version of $\mathbf{Y}$. Given random vectors $\mathbf{Y}$ and $\mathbf{Z}$ of dimension p, then $\mathbf{Y}$ induces (via $\mathbf{U}$) an order $\mathbf{UZ}$, called the induced or concomitant order. When the underlying probability model of $\mathbf{Y}$ is permutation-symmetric, that is, $\mathrm{F}_{\mathbf{y}}(\mathbf{y}) = \mathrm{F}_{\mathbf{y}}(\mathrm{g}\mathbf{y})$ for all permutation matrices g, then $\mathbf{U}$ is a random permutation matrix uniformly distributed in the class of $\mathrm{p} \times \mathrm{p}$ permutation matrices. To prove Proposition 2.1, we pointed to the fact that permutation-symmetric matrices commute with every permutation matrix of corresponding dimension, that is, if $\mathbf{\Sigma} = \alpha\mathbf{e}\mathbf{e}^{'} + \beta\mathbf{I}$ for scalars α and β, then $\mathrm{g}\mathbf{\Sigma} = \mathbf{\Sigma}\mathrm{g}$ for all permutation, g of same dimension as $\mathbf{\Sigma}$ (the converse being also true).

The case $\lambda_{12} = \gamma_{12}$ in Corollary 2.1 is of particular interest. This is the case when the covariance structure between $\mathbf{Y}$ and $\mathbf{Z}$ is proportional to $\mathbf{e}\mathbf{e}^{'}$. Then we have $\mathrm{Cov}(\mathbf{Y}, \mathbf{Z}) = \mathrm{Cov}(\mathcal{Y}, \mathcal{Z})$. Furthermore, $\mathrm{E}(\mathbf{Z}) = \mathrm{E}(\mathcal{Z})$ and $\mathrm{Cov}(\mathbf{Z}) = \mathrm{Cov}(\mathcal{Z})$ so that the parameter space in $\mathbf{Z}$ is not affected by ordering on the $\mathbf{Y}$ space.

A related result for two ordered multivariate normal variates is as follows: Suppose that the joint distribution of $(\mathbf{Y}_1, \mathbf{Y}_2)$ is multivariate normal satisfying the **BPS** condition, and $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are related by $\mathbf{Y}_2 = \mathbf{T}\mathbf{Y}_1 + \mathbf{V}$, where $\mathbf{Y}_1$ and $\mathbf{V}$ are independent and $\mathbf{T} = \mathbf{\Sigma}_{21}\mathbf{\Sigma}_{11}^{-1}$. Let $\mathcal{Y}_{\mathrm{i}}$ indicate the ordered version of $\mathbf{Y}_{\mathrm{i}}$, i=1,2. Then

$$\begin{aligned} \mathrm{Cov}[(\mathcal{Y}_1, \mathcal{Y}_2)] &= \mathbf{A}[\mathrm{Cov}[(\mathbf{Y}_1, \mathbf{V})] \otimes \mathcal{C}]\mathbf{A}' \\ &= \mathbf{A}\mathrm{Cov}[(\mathbf{Y}_1, \mathbf{V})]\mathbf{A}' \otimes \mathcal{C} \\ &= \mathrm{Cov}[(\mathbf{Y}_1, \mathbf{Y}_2)] \otimes \mathcal{C}, \end{aligned}$$

where $\mathbf{A}$ is given in (3) and $\otimes$ indicates the Kronecker product of block matrices. When $\lambda_{12} = \gamma_{12}$, using the fact that $\mathbf{\Sigma}_{12}\mathcal{C} = \mathbf{\Sigma}_{12}$, then $\mathrm{Cov}(\mathcal{Y}_1, \mathcal{Y}_2) = \mathrm{Cov}(\mathbf{Y}_1, \mathbf{Y}_2) = \mathbf{\Sigma}_{12}$, as shown in Viana and Olkin (1997).

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