Combined Bayesian Estimates for the Equicorrelation Coefficient

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Abstract

Combined Bayesian estimates for equicorrelation covariance matrices are considered. The case of a common equicorrelation ρ and possibly different standard deviations $\sigma_1, \ldots, \sigma_k$ among k experimental groups is examined first, and the Bayesian estimation of $(\rho, \sigma_1, \ldots, \sigma_k)$ is discussed. Secondly, under the assumption of a common standard deviation and possibly different equicorrelations, the Bayesian estimation of $(\rho_1, \ldots, \rho_k, \sigma)$ is considered.

1. Introduction

Symmetric covariance models are naturally appropriate to model corneal curvature maps (keratometry) $\mathbf{X}$, consisting of curvature measurements $X_1, \ldots, X_p$. Of particular interest is the equicorrelation model in which $\mathrm{Var}(X_1) = \ldots = \mathrm{Var}(X_p) = \sigma^2$ and ρ is the common correlation between curvatures of any two distinct measurement points. The objective of this paper is to obtain combined Bayesian estimates for the underlying covariance structure when samples of $\mathbf{X}$ are available from independent collaborating experiments.

An example of keratometry data is considered in the next section. The corresponding models, statistical analyses and applications are presented in Sections 3, 4 and 5, respectively. Additional comments and a summary are included in Section 6. Technical derivations are outlined in the Appendix.

Combined estimation of the the equicorrelation covariance structure is also important in applications dealing with bilateral units of analysis such as fellow eyes and fellow ears [Rosner (1982, 1984)], as well as in educational and psychological research (Olkin 1967). The model with common equicorrelation and possibly different variances among the k studies fits the general framework of a common principal components model discussed in detail by Flury (1988). Similarly, the same model is related to the general problem of estimating k covariance matrices with a common

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correlation matrix considered by Manly and Rayner (1987). A comprehensive discussion of methods of combining estimates of the standard correlation coefficient can be seen in Viana (1980), Hedges and Olkin (1985), and more recently in Han (1989).

2. The Keratometry Data

A typical curvature map $\mathbf{X}$ is based on about 20 concentric fields (rings or zones), which may be divided into 36 equally spaced semimeridians. The curvature is measured at each field and semimeridian so that each curvature mapping consists of 720 or more curvature measurements (Klyce and Wilson 1989, Hage 1976, Waring 1989). The present data are derived from $k = 3$ collaborating studies designed to assess a computerized corneal topographer and are described in detail in Viana, Olkin and McMahon (1993). In each experiment the topography of a calibrated steel ball is replicated $N = 10$ times and the results are considered as independent samples of $\mathbf{X}$.

Consider, specifically, the subregion of $\mathbf{X}$ specified by $p = 4$ semimeridans $19, 20, 21, 22$ at the aperture corresponding to field 18 in each collaborating experiment $i = 1, 2, 3$. The calibrated balls have curvatures, in diopters (D), 40.5 D, 42.5 D and 44.75 D. The observed correlation matrices R_i, and the vector $\mathbf{s}_i$ of sample standard deviations for experiment i, based on $N_i = 10$ repeated mappings are :

$$R_1 = \begin{bmatrix} 1.000 & 0.987 & 0.917 & 0.944 \\ 0.987 & 1.000 & 0.889 & 0.917 \\ 0.917 & 0.889 & 1.000 & 0.952 \\ 0.944 & 0.917 & 0.952 & 1.000 \end{bmatrix}, \quad \mathbf{s}_1' = (.061,\ .078,\ .049,\ .045),$$

$$R_2 = \begin{bmatrix} 1.000 & 0.748 & 0.983 & 0.958 \\ 0.748 & 1.000 & 0.833 & 0.796 \\ 0.983 & 0.833 & 1.000 & 0.962 \\ 0.958 & 0.796 & 0.962 & 1.000 \end{bmatrix}, \quad \mathbf{s}_2' = (.098,\ .138,\ .082,\ .086),$$

$$R_3 = \begin{bmatrix} 1.000 & 0.990 & 0.943 & 0.979 \\ 0.990 & 1.000 & 0.943 & 0.979 \\ 0.943 & 0.943 & 1.000 & 0.923 \\ 0.979 & 0.979 & 0.923 & 1.000 \end{bmatrix}, \quad \mathbf{s}_3' = (.124,\ .124,\ .144,\ .120).$$

We observe that within each matrix, there is considerable homogeneity among the correlations and among the variances. However, between refractive powers, the variability in the data seems to differ.

3. The Models

Suppose that a multivariate normal experiment on p random variables leads to the estimation of its covariance parameters under the equicorrelation structure

$$\Sigma = \mathrm{Cov}(\mathbf{X}) = \sigma^2 \begin{bmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \dots & \rho \\ \vdots & \vdots & \dots & \vdots \\ \rho & \rho & \dots & 1 \end{bmatrix}, \quad \frac{-1}{p-1} < \rho < 1. \tag{3.1}$$

The covariance matrix contributes with two parameters; the intraclass or equicorrelation ρ among any two distinct components of $\mathbf{X}$, and the variance parameter $\sigma^2 = \mathrm{Var}(X_j),\ j = 1, \dots, p$. The experiment is then replicated by k collaborating independent trials or studies, judged to be based on comparable experimental criteria. In this case, combined estimates of the joint covariance structure $(\rho_1, \sigma_1, \dots, \rho_k, \sigma_k)$ become appropriate. Two basic models are considered in this article: The **common-equicorrelation** model assumes that $\rho_1 = \dots = \rho_k$ so that the purpose is jointly estimating $\{\rho, \sigma_1, \dots, \sigma_k\}$ based on $S_1, \dots, S_k$, the corresponding sample covariance matrices. The **equal- variance equicorrelation** model assumes that $\sigma_1 = \dots = \sigma_k$ and the joint estimation of $\{\rho_1, \dots, \rho_k, \sigma\}$ is considered.

4. Bayesian Estimates

Consider k independent sample covariance matrices $S_1, \dots, S_k$ of dimension p based on sample sizes $N_1, \dots, N_k$ such that the distribution of $A_i = N_i S_i$ is Wishart with parameters Σ_i and $n_i = N_i - 1$, indicated by $A_i \sim W_p(\Sigma_i, n_i),\ i = 1, \dots, k$. Suppose also that each covariance matrix Σ_i has the equicorrelation form (3.1), with corresponding variance σ_i^2 and equicorrelation ρ_i. The subscript i indicates the experiments or studies being combined; its range is $1, \dots, k$ and will be repeated in the remaining of the paper only when necessary. The following results are derived in the Appendix.

4.1 Inferences under the common-equicorrelation model

Under the common-equicorrelation model there is a common equicorrelation parameter ρ and there are k posssibly distinct variance parameters σ_i^2 among the independent studies. The posterior distribution g of ρ with respect to $\prod_i [\sigma_i^{-2} d\sigma_i^2] d\rho$ is given by

$$g(\rho) \propto \frac{[1 + (p-1)\rho]^{(p-1)n/2}(1-\rho)^{n/2}}{\prod_i \{[1 + (p-1)\rho](p-1)(1-r_i) + (1-\rho)[1 + (p-1)r_i]\}^{pn_i/2}}, \tag{4.1}$$

where $\frac{-1}{p-1} < \rho < 1$, $n = \sum_i n_i$ and r_i is the sample equicorrelation coefficient

$$r_i = \frac{2\sum_{u<v}(S_i)_{uv}}{(p-1)trS_i}$$

observed in the i-th study. The posterior moments of order $h = 0, 1, \dots$ follow from numerically integrating $\rho^h g(\rho)$ in its (bounded) domain. The posterior mean and variance of σ_i^2 can be derived from the fact that σ_i^2 is the average of the p eigenvalues $\lambda_{i1} = \sigma_i^2[1+(p-1)\rho]$ and $\lambda_{ij} = \sigma_i^2(1-\rho), j = 2, \dots, p$. The corresponding expressions for the posterior mean and variance of σ_i^2 are shown in the Appendix [(A.9),(A.10)].

4.2 Inferences under the common-variance equicorrelation model.

Under the common-variance equicorrelation model there is a common variance parameter σ^2 and there are posssibly k distinct equicorrelation parameters ρ_i among the collaborating studies. The joint posterior distribution g of ρ_i with respect to $\sigma^{-2}d\sigma^2\prod_i d\rho_i$ is given by

$$g(\rho_1,\ldots,\rho_k) \propto \frac{\prod_i\{[1+(p-1)\rho_i]^{-n_i/2}(1-\rho_i)^{-(p-1)n_i/2}\}}{\{\sum_i \omega_i[(p-1)\frac{1-r_i}{1-\rho_i}+\frac{1+(p-1)r_i}{1+(p-1)\rho_i}]\}^{pn/2}}, \tag{4.2}$$

where $\frac{-1}{p-1} < \rho_i < 1$, r_i is the sample equicorrelation coefficient observed in the i-th study, and $\omega_i = \mathrm{tr}A_i/\sum_u trA_u$. The posterior mixed-moments of order $h_1,\ldots,h_k$ follow from numerically evaluating the necessary versions of $\int \rho_1^{h_1}\cdots\rho_k^{h_k}g(\rho_1,\ldots,\rho_k)\prod_i d\rho_i$ in its (bounded) domain. Combined inferences for σ^2 based on the posterior distribution of λ_{i1} and λ_{i2} are discussed in the Appendix [(A.9), (A.10) and (A.12)].

5. Analysis of the Keratometry Data

In the present case there are $k = 3$ experiments, each one consisting of observing $N_i = 10$ curvature mappings $\mathbf{X}$ of the anterior reflective surface of a calibrated steel ball of different curvature. Repeated sample estimates of Σ_i in our study, for several subregions of any given concentric field, have suggested an equicorrelation structure (σ_i^2, ρ_i) with common equicorrelation $\rho = \rho_i$, and possibly different variances σ_i^2, corresponding to different refractive powers among steel balls (common-equicorrelation model). Consequently, the parameters of interest are $(\rho, \sigma_1, \sigma_2, \sigma_3)$, subject to $k = 3$ and $N_i = 10$. The data presented in Section 2 is based on observed curvatures at $p = 4$ consecutive semimeridians $\{19, 20, 21, 22\}$ at the aperture corresponding to field 18. Note that the unrestricted number of covariance parameters within each experiment is $[p(p-1)/2] + p = 10$, compared with 2, under the equicorrelation model.

Table 1 shows the component sample estimates $s_{i1}, s_{i2}, s_{i3}, s_{i4}$ of σ_i, the standard deviation of curvature measurements at semimeridians $\{19, 20, 21, 22\}$, and sample estimates r_i of the common equicorrelation ρ, for each calibrated steel balls $i = 1, 2, 3$, based on samples of size $N_i = 10$.

A standard numerical integration procedure showed that the the posterior estimate of ρ based on (4.1) is

$$E(\rho \mid r_1, r_2, r_3) = 0.939,$$

indicating a fairly large redundancy among the observed curvatures within the selected subset of 4 consecutive semimeridians. In particular, $E(\rho^2 \mid r_1, r_2, r_3) = 0.883$ and

$$Sd(\rho \mid r_1, r_2, r_3) = 0.022$$

is the posterior standard deviation of the combined estimate of ρ. Numerical evaluation of the posterior cumulative distribution of ρ showed that ρ falls in the interval (0.88, 0.97) with probability 0.96. Figure 1 illustrates the posterior distribution of ρ. Table 2 shows the posterior estimates of σ_i^2 based on the posterior estimates of the eigenvalues of Σ_i. Expressions (A.9) and (A.10) were used to derive the results. The confidence limits for σ_i^2 shown in Table 2 are based on 1.5 Sd.

6. Summary and Discussion

We have derived the combined posterior estimates of the equicorrelation parameter when sample covariance matrices $S_1, \ldots, S_k$ are available from several independent collaborating studies or experiments.

The combined posterior estimate of ρ under the common-equicorrelation model depends on the available data only through the sample equicorrelations and their corresponding sample sizes. The posterior distribution of ρ is given by (4.1). Standard numerical methods for integration in a given bounded interval can be used to determine the posterior moments of ρ. In applications with $p = 2$ [e.g. vision data] the posterior distribution of ρ becomes

$$g(\rho) \propto \frac{(1-\rho^2)^{n/2}}{\prod_i (1-\rho r_i)^{n_i}}.$$

The joint posterior estimate of $(\rho_1, \ldots, \rho_k)$ under the equal- variance equicorrelation model depend of $r_1, \ldots, r_k$, their sample sizes and the weigths $\omega_i = \mathrm{tr}A_i / \sum_u \mathrm{tr}A_u$. When $p = 2$ the joint posterior distribution (4.2) becomes

$$g(\rho_1, \ldots, \rho_k) \propto \frac{\prod_i (1-\rho_i^2)^{-n_i/2}}{\{\sum_i \omega_i \frac{1-\rho_i r_i}{1-\rho_i^2}\}^n}. \tag{6.1}$$

Standard numerical methods for integration in a given bounded region can be used to determine the posterior mixed moments of $\rho_1, \ldots, \rho_k$.

The equal-variance equicorrelation model was also applied to the Keratometry data discussed earlier. Table 3 shows the posterior marginal estimates of ρ_i and corresponding standard deviations and credibility limits. Figure 2 shows the cross-sections g_1, g_2, g_3 of the joint posterior distribution when all but one component are fixed and equal to the corresponding posterior marginal mean $\hat{\rho}_i$. For example, $g_1(\rho) = g(\rho, \widehat{\rho_2}, \widehat{\rho_3})$

Combined maximum likelihood estimation of the equicorrelation structure was recently reported by the author (Viana 1994). The combined maximum likelihood estimate of ρ under the common equicorrelation model $(\rho, \sigma_1, \ldots, \sigma_k)$ and the corresponding Bayesian estimates depend on the available data only through the sample intraclass correlations $r_i \quad i = 1, \ldots, k$, and their relative sample sizes. The MLE of ρ is the unique solution of a polynomial equation of degree k in its domain. Standard numerical methods for determining the zeros of an equation in a given bounded interval can be used to determine the solution $\widehat{\rho}$. However, the estimated variance of $\widehat{\rho}$ is restricted to large-sample approximations. In contrast, the posterior moments of order $h = 0, 1, \ldots$ follow from numerically integrating $\rho^h g(\rho)$ in its (bounded) domain, as described earlier (4.1). Moreover, the exact posterior mean and variance of σ_i^2 can also be obtained directly from (A.9) and (A.10).

The joint MLE of $(\rho_1, \ldots, \rho_k)$ and corresponding Bayesian joint estimates under the equal-variance equicorrelation model $(\rho_1, \ldots, \rho_k, \sigma)$ depend on $r_1, \ldots, r_k$, their relative sample sizes and the relative weights $\omega_i = \mathrm{tr}A_i / \sum_{j=1}^k \mathrm{tr}A_j, \; i = 1, \ldots, k$. The joint MLE is the numerical iterative solution of a system of polynomial equations and the corresponding variances are restricted to large-sample approximations, whereas, in contrast, the posterior mixed-moments of order $h_1, \ldots, h_k$ shown in (4.2) follow from numerically evaluating the necessary versions of $\int \rho_1^{h_1} \cdots \rho_k^{h_k} g(\rho_1, \ldots, \rho_k) \prod_i d\rho_i$ in its (bounded) domain. Consequently, when the sample sizes associated with the collaborating studies are small to moderate, the Bayesian method described above can be of particular advantage.

The estimates discussed above are based on the covariance estimates alone. However, when mean estimates are also considered, the additional assumption of symmetry $\mu_{ij} = m_i$ among the components of the underlying vector of means $\mu_i = (\mu_{i1}, \ldots, \mu_{ip})$ suggests the neccesary modifications to derive the combined estimates. Inferences about the two distinct eigenvalues remain essentially the same. The posterior distribution of λ_{i1} remains exactly the same, and the posterior distribution of λ_{i2} now has $v_{i2} = (p-1)N_i/2$ degrees of freedom and is based on $t_{i2} + N_i \sum_j (\bar{x}_{ij} - \bar{x}_i)^2/2$. The posterior distribution of m_i can also be obtained and expressed in terms of a t distribution. However, the posterior distribution of the equicorrelation coefficients ρ_i cannot be shown in explicit form without the presence of the parameter m_i.

The posterior estimate of a common ρ under the assumption of symmetric multivariate normal distributions is of particular interest to interpret familial data. Iterative maximum likelihood estimates have been considered by several authors including Donner and Koval (1980), Donner (1986), Donner and Bull (1983), Shoukri and Ward (1984), Srivastava (1984), Srivastava, Keen and Katapa (1988), Paul (1990), Paul and Barnwal (1990) and Gleser (1992). The posterior estimates of the equicorrelation coefficient derived in this paper can easily be aplicable to familial data as well. Of particular interest is the common- equicorrelation model based on k independent sample covariance matrices of possibly different orders $p_1, \ldots, p_k$, to account for families with different number of members. In that case, the posterior distribution (4.1) of the common equicorrelation parameter becomes

$$g(\rho) \propto \frac{(1-\rho)^{n/2} \prod_i [1 + (p_i - 1)\rho]^{(p_i-1)n_i/2}}{\prod_i \{[1 + (p_i - 1)\rho](p_i - 1)(1 - r_i) + (1 - \rho)[1 + (p_i - 1)r_i]\}^{p_i n_i/2}}, \tag{6.2}$$

where $\min_i \{\frac{-1}{p_i - 1}\} < \rho < 1$. Although the present estimates are based on sample covariance data alone, the method applies to small samples and posterior credibility intervals can be obtained using standard methods of numerical intergration in bounded regions.

A. Appendix

Let $A_i \sim W_p(\Sigma_i, n_i),\ i = 1, \ldots, k$ where Σ_i has the equicorrelation form (3.1), with corresponding variance σ_i^2 and equicorrelation ρ_i. If Γ is an orthogonal $p \times p$ matrix with first column constant and equal to $1/\sqrt{p}$ [e.g. Press (1972, pages 13,14)], then $A_i^* = \Gamma' A_i \Gamma \sim W_p(D_i, n_i)$, where D_i is a $p \times p$ diagonal matrix with diagonal entries

$$(D_i)_{11} = \lambda_{i1} = \sigma_i^2 [1 + (p-1)\rho_i], \qquad (D_i)_{jj} = \lambda_{i2} = \sigma_i^2 (1 - \rho_i), \quad j = 2, \ldots, p. \tag{A.1}$$

Consequently, the likelihood $\prod_i L(\sigma_i, \rho_i)$ can be expressed in terms of $\prod_{i=1}^k L(\lambda_{i1}, \lambda_{i2})$, which factors orthogonally into $\prod_i P(q_{i1} \mid \lambda_{i1}) \prod_{j=2}^p P(q_{ij} \mid \lambda_{i2})$, where $P(\cdot \mid \cdot)$ indicates the appropriate density function corresponding to

$$Q_{i1} = (A_i^*)_{11} = \frac{\sum_{u,v} (A_i)_{uv}}{p} \sim W_1(\lambda_{i1}, n_i), \quad Q_{ij} = (A_i^*)_{jj} \sim W_1(\lambda_{i2}, n_i), \quad j = 2, \ldots, p.$$

Also note that $T_{i1} = Q_{i1}/2$ can be expressed as

$$T_{i1} = \frac{N_i}{2}[1 + (p-1)r_i]trS_i/p \sim W_1(\frac{\lambda_{i1}}{2}, n_i), \tag{A.2}$$

where r_i is the conventional estimate of the equicorrelation coefficient ρ_i (Olkin 1967, Olkin and Pratt 1958, Olkin and Siotani 1964),

$$r_i = \frac{2\sum_{u<v}(S_i)_{uv}}{(p-1)trS_i}. \tag{A.3}$$

Also, $Q_{i1'} = \sum_{j=2}^{p} Q_{ij} = \mathrm{tr}(\Gamma' A_i \Gamma) - (\Gamma' A_i \Gamma)_{11}$, and since the matrix Γ is orthogonal,

$$T_{i2} = \frac{1}{2}\{trA_i - \frac{1}{p}\sum_{u,v}(A_i)_{uv}\} = \frac{N_i}{2}(p-1)(1-r_i)trS_i/p \sim W_1(\frac{\lambda_{i2}}{2}, (p-1)n_i). \tag{A.4}$$

Inferences on λ_{i1}. Let $v_{i1} = n_i/2$. From (A.2), the likelihood function based on T_{i1} can be expressed in terms of $\gamma_1(v_{i1}, \lambda_{i1}^{-1})$, a gamma-1 distribution with parameters v_{i1} and λ_{i1}^{-1}, so that the posterior distribution

$$g(\lambda_{i1} \mid t_{i1}) \propto f_{\gamma 1}(t_{i1} \mid v_{i1}, \frac{1}{\lambda_{i1}})\frac{1}{\lambda_{i1}}, \tag{A.5}$$

of λ_{i1} with respect to λ_{i1}^{-1} is an inverted-gamma-1 distribution $\gamma_{inv,1}(d, y)$ with parameters v_{i1} and t_{i1}, [e.g. (Raiffa and Schlaifer 1968, pg. 227)]. In particular,

$$E(\lambda_{i1} \mid t_{i1}) = \frac{t_{i1}}{v_{i1}-1}, \qquad \mathrm{Var}(\lambda_{i1} \mid t_{i1}) = \frac{t_{i1}^2}{(v_{i1}-1)^2(v_{i1}-2)}. \tag{A.6}$$

Inferences on λ_{i2}. Let $v_{i2} = (p-1)n_i/2$. The likelihood $L(\lambda_{i2})$ depends only on $Q_{ij} \sim W_1(\lambda_{i2}, n_i)$, $\quad j = 2, \ldots, p$, which, given λ_{i2}, are independent. Similarly to λ_{i1}, from (A.4) it follows that $T_{i2} \mid \lambda_{i2} \sim \gamma_1(v_{i2}, \lambda_{i2}^{-1})$, and $\lambda_{i2} \mid t_{i2} \sim \gamma_{inv,1}(v_{i2}, t_{i2})$, with respect to $p(\lambda_{i2}) \propto \lambda_{i2}^{-1}$, independent of the posterior distribution of λ_{i2}, with respect to $\lambda_{i1}^{-1}\lambda_{i2}^{-1}$. In particular,

$$E(\lambda_{i2} \mid t_{i2}) = \frac{t_{i2}}{v_{i2}-1}, \qquad \mathrm{Var}(\lambda_{i2} \mid t_{i2}) = \frac{t_{i2}^2}{(v_{i2}-1)^2(v_{i2}-2)}. \tag{A.7}$$

Posterior distribution of ρ_i. Let $l_i = 1-\rho_i$, $u_i = 1 + (p-1)\rho_i$. From (A.2) and (A.4) it follows that the posterior distribution of ρ_i with respect to σ_i^{-2} is

$$g(\rho_i \mid t_{i1}, t_{i2}) \propto \int f_{\gamma_1}(t_{i1} \mid v_{i1}, \frac{1}{u_i\sigma_i^2}) f_{\gamma_1}(t_{i2} \mid v_{i2}, \frac{1}{l_i\sigma_i^2})\frac{1}{\sigma_i^2}d\sigma_i^2 \propto \frac{l_i^{v_{i1}}u_i^{v_{i2}}}{\{l_i t_{i1} + u_i t_{i2}\}^{v_{i1}+v_{i2}}}. \tag{A.8}$$

Since $N_i trS_i/2 = t_{i1} + t_{i2}$ can be factored out of $\{l_i t_{i1} + u_i t_{i2}\}$ and is determined by the data $\{t_{i1}, t_{i2}\}$ alone, the posterior distribution of ρ_i can be expressed as

$$g(\rho_i \mid t_{i1}, t_{i2}) \propto \frac{[1+(p-1)\rho_i]^{(p-1)n_i/2}(1-\rho_i)^{n_i/2}}{\{[1+(p-1)\rho_i](p-1)(1-r_i) + (1-\rho_i)[1+(p-1)r_i]\}^{pn_i/2}}.$$

Inferences on σ_i^2. To estimate σ_i^2 we use the fact that $p\sigma_i^2 = \lambda_{i1} + (p-1)\lambda_{i2}$. Consequently, the posterior estimate of σ_i^2 with respect to $(\lambda_{i1}\lambda_{i2})^{-1}$ is, from (A.6) and (A.7),

$$E(\sigma_i^2 \mid t_{i1}, t_{i2}) = [\frac{t_{i1}}{v_{i1}-1} + (p-1)\frac{t_{i2}}{v_{i2}-1}]/p. \tag{A.9}$$

Because λ_{i1} and λ_{i2} are independent, it follows that

$$\mathrm{Var}(\sigma_i^2 \mid t_{i1}, t_{i2}) = [\frac{t_{i1}^2}{(v_{i1}-1)^2(v_{i1}-2)} + (p-1)^2 \frac{t_{i2}^2}{(v_{i2}-1)^2(v_{i2}-2)}]/p^2. \tag{A.10}$$

Inferences under the common-equicorrelation model. Under the common-equicorrelation model there is a common value l of $l_i = 1-\rho_i$ and a common value u of $u_i = 1+(p-1)\rho_i$. Let $n = \sum n_i$, $v_1 = \sum v_{i1} = n/2$, $v_2 = \sum v_{i2} = (p-1)n/2$, $t_1 = \{t_{11}, \ldots, t_{k1}\}$, $t_2 = \{t_{12}, \ldots, t_{k2}\}$. The posterior distribution of the common equicorrelation ρ given the joint data t_1, t_2 with respect to $[\prod \sigma_i^2]^{-1}$ is, from (A.8),

$$g(\rho \mid t_1, t_2) \propto \prod_i \int f_{\gamma_1}(t_{i1} \mid v_{i1}, \frac{1}{u\sigma_i^2}) f_{\gamma_1}(t_{i2} \mid v_{i2}, \frac{1}{l\sigma_i^2}) \frac{1}{\sigma_i^2} d\sigma_i^2 \propto \frac{l^{v_1} u^{v_2}}{\prod_i \{l_i t_{i1} + u_i t_{i2}\}^{v_{i1}+v_{i2}}},$$

which can be expressed as

$$g(\rho \mid t_1, t_2) \propto \frac{[1+(p-1)\rho]^{(p-1)n/2}(1-\rho)^{n/2}}{\prod_i \{[1+(p-1)\rho](p-1)(1-r_i) + (1-\rho)[1+(p-1)r_i]\}^{pn_i/2}}.$$

Inferences for σ_i^2 based on the posterior distribution of λ_{i1} and λ_{i2} follow without change from (A.9) and (A.10).

Inferences under the common-variance equicorrelation model. Under the common-variance equicorrelation model there is a common value σ of σ_i^2. The joint posterior distribution of $\rho_1, \ldots, \rho_k$ given t_1, t_2 with respect to σ^{-2} follows from observing that (A.8) can be expressed as

$$g(\rho_i \mid t_{i1}, t_{i2}) \propto \frac{l_i^{-v_{i2}} u_i^{-v_{i1}}}{\{\frac{t_{i1}}{u_i} + \frac{t_{i2}}{l_i}\}^{v_{i1}+v_{i2}}}, \tag{A.11}$$

which then generalizes to

$$g(\rho_1, \ldots, \rho_k \mid t_1, t_2) \propto \int \prod_i f_{\gamma_1}(t_{i1} \mid v_{i1}, \frac{1}{u_i\sigma^2}) f_{\gamma_1}(t_{i2} \mid v_{i2}, \frac{1}{l_i\sigma^2}) \frac{1}{\sigma^2} d\sigma^2 \propto \frac{\prod_i l_i^{v_{i2}} u_i^{v_{i1}}}{\sum_i \{\frac{t_{i1}}{u_i} + \frac{t_{i2}}{l_i}\}^{v_1+v_2}},$$

and can be expressed as

$$g(\rho_1, \ldots, \rho_k \mid t_1, t_2) \propto \frac{\prod_i \{[1+(p-1)\rho_i]^{-n_i/2}(1-\rho_i)^{-(p-1)n_i/2}\}}{\{\sum_i \frac{N_i}{2} trS_i[(p-1)\frac{1-r_i}{1-\rho_i} + \frac{1+(p-1)r_i}{1+(p-1)\rho_i}]\}^{pn/2}},$$

where $n = \sum_i n_i$. Note that in the above expression the coefficients $N_i trS_i/2$ are determined by t_1, t_2 and can be replaced by the weights $\omega_i = trA_i / \sum_u trA_u$.

Combined inferences for σ^2 based on the posterior distribution of λ_{i1} and λ_{i2} follow from (A.9) and (A.10), observing that $\sigma^2 = \sum_i [\lambda_{i1} + (p-1)\lambda_{i2}]/kp$ and that $\{\lambda_{i1}, \lambda_{i2}, i = 1, \ldots, k\}$ are jointly independent, given $\{t_1, t_2\}$. Consequently, the mean combined estimate of σ^2 and its variance are

$$E(\sigma^2 \mid t_1, t_2) = \frac{1}{k}\sum_i E(\sigma_i^2 \mid t_{i1}, t_{i2}), \qquad \mathrm{Var}(\sigma^2 \mid t_1, t_2) = \frac{1}{k^2}\sum_i \mathrm{Var}(\sigma_i^2 \mid t_{i1}, t_{i2}). \tag{A.12}$$

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Figure 1: Posterior distribution of the equicorrelation coefficient under the common-equicorrelation model.

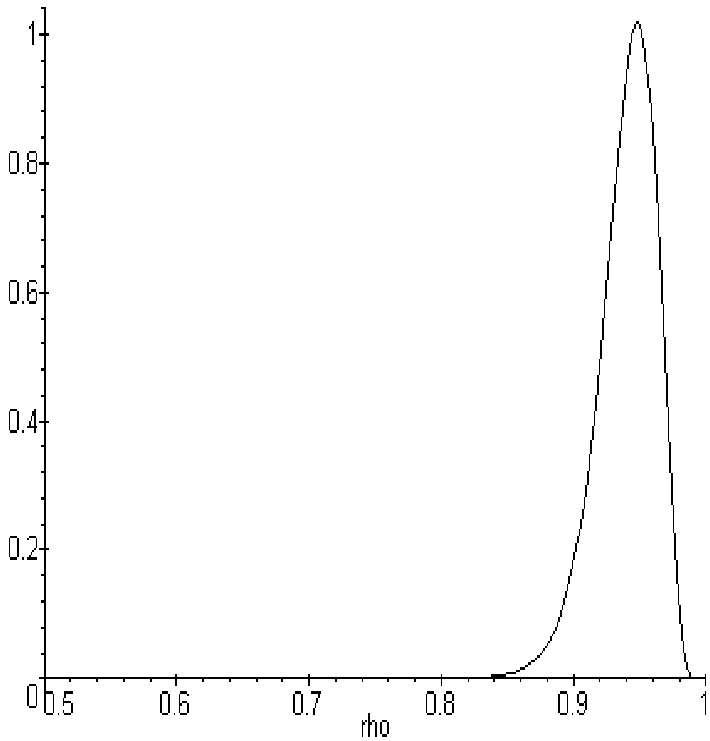

Figure 2: Cross-sections g_1, g_2, g_3 of the joint posterior distribution when all but one component are fixed and equal to the corresponding posterior marginal mean $\hat{\rho}_i$ of ρ_i.

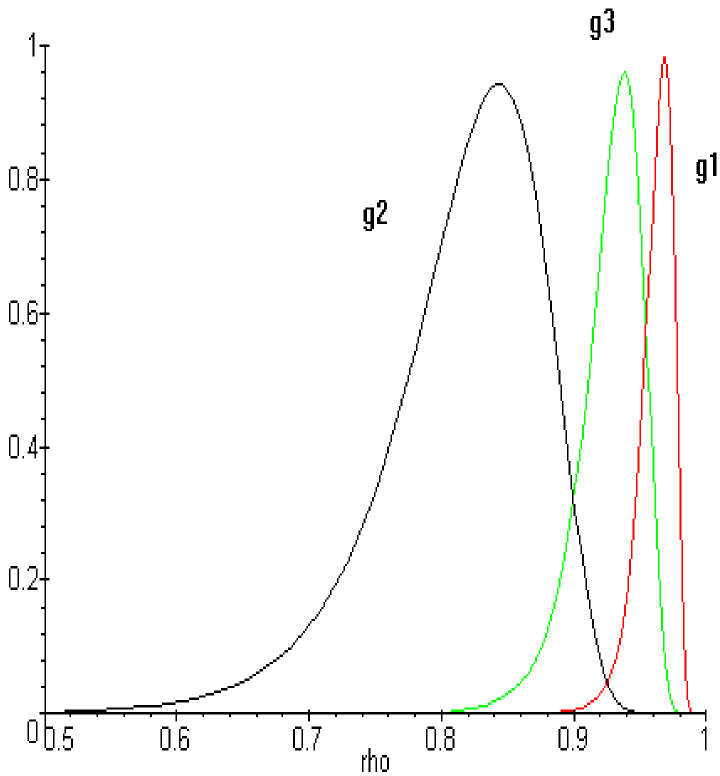

Table 1: Component sample estimates $s_{i1}, s_{i2}, s_{i3}, s_{i4}$ of σ_i, the standard deviation of curvature measurements, and sample estimate r_i of the common equicorrelation ρ, for each calibrated steel ball $i = 1, 2, 3$ at semimeridians $\{19, 20, 21, 22\}$.

Ball [i]	Curvature	N_i	s_{i1}	s_{i2}	s_{i3}	s_{i4}	r_i
1	40.50	10	.061	.078	.049	.045	.879
2	42.50	10	.098	.138	.089	.086	.814
3	44.75	10	.124	.124	.144	.120	.952

Table 2: Posterior estimates of σ_i^2 under the common-equicorrelation model, for each calibrated steel balls $i = 1, 2, 3$ at semimeridians $\{19, 20, 21, 22\}$.

i	t_{i1}	t_{i2}	$E(\lambda_{i1}$	$Sd(\lambda_{i1})$	$E(\lambda_{i2})10^4$	$Sd(\lambda_{i2})10^4$	$E(\sigma_i^2)$	$Sd(\sigma_i^2)$	L_i	U_i
1	.0654	.0065	.0187	.0118	5.2272	1.5414	.0050	.0029	.0006	.0095
2	.1893	.0306	.0540	.0342	24.5520	7.2399	.0153	.0085	.0025	.0282
3	.3181	.0118	.0908	.0574	9.5040	2.8025	.0234	.0143	.0018	.0449

Table 3: Posterior estimates of ρ_i and corresponding credibility limits under the common-variance equicorrelation model, for each calibrated steel balls $i = 1, 2, 3$ at semimeridians $\{19, 20, 21, 22\}$.

Ball[i]	r_i	$E(\rho_i)$	$E(\rho_i^2)$	$Sd(\rho_i)$	L_i	U_i	credibility
1	.879	.963	.928	.0113	.93	.98	.92
2	.814	.825	.685	.0661	.66	.91	.92
3	.952	.930	.866	.0331	.89	.96	.81