Dihedral representations and statistical geometric optics. I. Spherocylindrical lenses

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Received September 22, 2004; revised manuscript received January 28, 2005; accepted March 16, 2005

The linear 2-dim irreducible representations of the dihedral groups (D_n) are interpreted as classical linear operators of geometrical optics. It is shown that the 2-dim irreducible representation of D_4 is simply the refractive group described by Campbell [Optom. Vision Sci. **74**, 381 (1997)]. The dihedral Fourier-inverse mechanism is introduced and shown to provide a systematic connection between the standard refractive data and their vector space representation, as proposed by Thibos *et al.* [Vision Sci. Appl. **2**, 14 (1994)]. © 2005 Optical Society of America

OCIS codes: 000.3860, 000.3870, 080.0080, 030.6600, 080.2720, 170.4460.

1. INTRODUCTION

Symmetry can be utilized in optical systems by using the algebraic techniques of group theory.[1,2] The fundamental relationship of linear optics is that for a ray passing through any optical system, in such a way that if ξ_0 is the incident vector and ξ is the emergent vector, then the two are related by a ray-transfer matrix that encapsulates all linear optical characteristics of the system. This linear-matrix-transfer approach has been extended to the nonlinear regime.[3,4] In the linear approximation, the ray-transfer matrix is a symplectic matrix.[5,6] In fact, one can even extend the analysis to show that the evolution of a light wave along an optical system can be described in terms of a group of operators acting as canonical transformations.[7] This group has a matrix representation realized by the symplectic group $S_p(4)$. Moreover,[8] Fourier transforms are elements of the so-called metaplectic group $M_p(2)$, which is isomorphic to $S_p(2)$. Therefore, the group of functional operators used in Fourier optics can be represented by a group of symplectic matrices of order 2 or 4. In this paper we utilize dihedral symmetries to systematically unify many theoretical, computational, and data analytical aspects present in linear geometrical optics.

Applications of group-theoretic principles in statistics and probability have a long history and tradition of their own.[9–15] The present paper introduces these principles in the context of the analysis of data from geometric optics using the techniques of symmetry studies for structured data introduced by one of the authors.[16,17] Briefly, structured data ($\mathbf{x}$) are data that are indexed by a set V of indices or labels (s) upon which certain symmetry relations can be consistently defined with the purpose of simplifying the structure and the analysis of the underlying data. Examples of simple structures are the set $V=C^L$ of all mappings $s:L\rightarrow C$, where L and C are finite sets; the set product $V=L\times C$, and its version $V=L\times\Omega$ in polar coordinates. Data from biological sequences are typically indexed by the set C^L and data from elementary experimental designs are often indexed by $L\times C$, whereas corneal surface curvature data and Shack–Hartmann wave-front sensor data may be indexed by a structure V of the type $L\times\Omega$. The labels in $L\times\Omega$ are at the intersection of concentric rings and equally spaced semimeridians, and provide the index for a surface curvature or for a point-spread function value. In particular applications, the data are indexed by an algebraic structure, such as a group (G) of symmetries. In this case, $V=G$. The refractive group[18] to be discussed below is an example of a structure for refractive data.

Symmetry studies take advantage of symmetries that are consistent with the set of labels to facilitate the classification, interpretation, and statistical analysis of the data $\mathbf{x}=\{x(s),\ s\in V\}$ indexed by these labels. These symmetries (σ) when consistently applied to the labels in V (in the sense of a group action) reduce the set as a disjoint union $V=\mathcal{O}_1\cup\ldots\cup\mathcal{O}_n$ of similarity orbits ($\mathcal{O}$). The action of G on V, in turn, leads to a linear representation (ρ) in the data vector space ($\mathcal{V}$). The resulting factorization $\mathcal{V}=\mathcal{V}_1\oplus\ldots\oplus\mathcal{V}_h$ of $\mathcal{V}$ is the consequence of defining a set of algebraically orthogonal projections ($\mathcal{P}$) that are linear combinations of these linear representations with real or complex scalar coefficients (the characters of the irreducible representations of G). More specifically, if G has h irreducible representations, then there are h projections, and the identity operator I in (each of the orbit-defined subspaces of) $\mathcal{V}$ reduces according to a sum

$$I = \mathcal{P}_1 + \mathcal{P}_2 + \cdots + \mathcal{P}_h \tag{1}$$

of these orthogonal projection matrices. Moreover, $\mathcal{P}_i \mathcal{P}_j = 0$ for $i \neq j$ and $\mathcal{P}_i^2 = \mathcal{P}_i$, $i = 1, \ldots, h$. These (canonical) decompositions are well known in the elementary theory of linear representations[19] and are routinely used in quantum chemistry, for example, in the determination of molecular bonding.[20] The connection with the data analytical component of any symmetry study follows from the observation that the basic decomposition

$$\|\mathbf{x}\|^2 = (\mathbf{x}|\mathbf{x}) = (\mathbf{x}|\mathcal{P}_1\mathbf{x}) + (\mathbf{x}|\mathcal{P}_2\mathbf{x}) + \cdots + (\mathbf{x}|\mathcal{P}_h\mathbf{x}) \tag{2}$$

for the sum of squares of the components of the data vector $\mathbf{x}$ for a particular inner product $(\cdot|\cdot)$ of interest (e.g., Euclidean, Hermitian, symplectic) can then always be obtained.

The canonical decomposition, we remark, establishes the systematic and unifying connection among the symmetries consistent with the labels, the structured data, and statistical inference. In fact, the statistical (Fisher–Cochran) theory of quadratic forms[21] can be readily applied to obtain new forms of analysis of variance, within which symmetry-related hypotheses can be identified and assessed.

In several experiments, however, the data are naturally indexed by a group of symmetries (the case $V = G$ mentioned above). For example, in many symmetry perception studies[22,23] the data are naturally indexed by rotational and axial symmetries. Similar experiments are potentially useful to describe and suggest interpretations to the (rotational, axial), symmetries present in human visual field data or in two-dimensional wave-front aberration data from the Shack–Hartmann wave-front sensor.[24] In this case, there is a one-to-one correspondence between the experimental data $\{x(\tau), \tau \in G\}$ and the Fourier transforms

$$\hat{x}(\beta) = \sum_{\tau \in G} x(\tau)\beta(\tau), \tag{3}$$

over the irreducible representations (β) of G. When the dimension of β is 2, for example, any 2×2 linear operator with real or complex coefficients can be uniquely decomposed in the form $\Sigma_{\tau \in G} x(\tau)\beta(\tau)$, with respect to the family $\{\beta(\tau); \tau \in G\}$ of elementary operators. One such family is precisely the so-called refractive group.[18]

Section 2 is a brief review of the symmetry operations defined by the dihedral group (D_n). These are the symmetries induced by the rotations and axial reflections of an n-sided regular polygon. The dihedral group is the basis of the applications discussed in this paper. In Section 3 we derive the canonical decomposition [Eq. (1)] and corresponding analysis of variance [Eq. (2)] for data that are indexed by the dihedral group D_4. In Section 4 we illustrate the application of the dihedral Fourier analysis [Eq. (3)] in the context of statistical geometric optics. It is our purpose to show, as previously suggested by one of the authors *et al.*,[25] that group theoretical analyses may allow newer methods of analysis of data consistent with certain symmetries of a given problem.

2. DIHEDRAL SYMMETRIES

In the present paper we consider the group (D_n) of symmetries of the n-sided regular polygon. These $2n$ symmetries are simply the n planar rotations by $2\pi/n$ deg and the n axial dihedral reflections leaving the polygon fixed. The potential significance of D_n in optics is that it can be represented in two-dimensional space with arbitrarily many levels of resolution. For example, for $n \geqslant 4$ even, there are $(n/2) - 1$ irreducible representations realized as planar rotations,

$$\beta_j(\eta^k) = \begin{bmatrix} \cos(w_{jk}) & -\sin(w_{jk}) \\ \sin(w_{jk}) & \cos(w_{jk}) \end{bmatrix}, \tag{3a}$$

and planar axial reflections

$$\beta_j(\eta^k \tau) = \begin{bmatrix} \cos(w_{jk}) & \sin(w_{jk}) \\ \sin(w_{jk}) & -\cos(w_{jk}) \end{bmatrix}, \tag{3b}$$

where $w_{jk} = 2\pi jk/n$, $j = 1, \ldots, (n/2) - 1$ and $k = 0, \ldots n - 1$. There are, in addition, n irreducible representations.[26]

The single 2-dim representation of D_4 has four planar rotations,

$$\beta(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \beta(\eta) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$\beta(\eta^2) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \beta(\eta^3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4}$$

respectively, the 0, 90, 180, 270 deg (clockwise) rotations, and four planar axial symmetries,

$$\beta(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \beta(\eta\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\beta(\eta^2\tau) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \beta(\eta^3\tau) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \tag{5}$$

corresponding to the horizontal, 45 deg line, vertical, and 135 deg axis, respectively. The table of irreducible characters (defined by the trace of the irreducible representations) of D_4 is

$$\left[\begin{array}{c|cccccccc} \chi & 1 & \eta & \eta^2 & \eta^3 & \tau & \eta\tau & \eta^2\tau & \eta^3\tau \\ \hline \chi_1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \chi_3 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ \chi_4 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ \chi_5 & 2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \end{array}\right]. \tag{6}$$

For example, the character χ_5 evaluates as the trace of the 2-dim irreducible representation β for D_4 described above.

We observe the following: The so-called refractive group[18] that appears in visual optics is exactly the 2-dim irreducible representation of D_4. The 2-dim irreducible representation of D_n in the linear space is unitarily equivalent to the representation

$$\tilde{\beta}_j(\eta^k) = \begin{bmatrix} \omega^{kj} & 0 \\ 0 & \bar{\omega}^{kj} \end{bmatrix}, \qquad \tilde{\beta}_j(\eta^k \tau) = \begin{bmatrix} 0 & \omega^{kj} \\ \bar{\omega}^{k} & 0 \end{bmatrix},$$

$k=0,\ldots n-1$, $j=1,\ldots,(n/2)-1$, in rotation space, where $\omega=\exp(2\pi i/n)$. A family of unitary transformations connecting the two representations is given by $X^{-1}\tilde{\beta}(\sigma)X=\beta(\sigma)$, $\sigma\in D_n$, with $X=\xi\left[\begin{smallmatrix} 1 & i \\ 1 & -i \end{smallmatrix}\right]$, up to a unimodular rotation factor $\xi\in\mathbb{C}$. It is of interest to note that the roots of the unit or the identity operator can be used to define and implement optical transformations that combine Fresnel diffraction and lens transformations.[27] From the unitary equivalence above, it follows that compensators and rotators of phase ω are simply unitarily equivalent 2-dim representations of D_n.

We observe that the cyclic group C_n is always a subgroup of D_n and that the dihedral group also appears as the product of the cyclic groups C_n and C_2, with a multiplication rule defined by

$$(\sigma,\tau)\times_{\alpha}(\sigma',\tau') = \begin{cases} (\sigma\sigma',\tau') & \text{when } \tau=1 \\ (\sigma\sigma'^{-1},\tau\tau') & \text{when } \tau=(12) \end{cases},$$

where (σ,τ) and (σ',τ') are in $C_n\times C_2$. This product $(\times_{\alpha})$ is called the semidirect product of C_n and C_2 and is defined by the group homomorphism $\alpha_{\tau}(\sigma)=\sigma$ if $\tau=1$, and $\alpha_{\tau}(\sigma)=\sigma^{-1}$ if $\tau\neq 1$, from C_2 to $\mathrm{Aut}(C_n)$.

3. DIHEDRAL DATA INVARIANTS AND CANONICAL ANALYSIS

In this section we will derive the canonical decomposition [Eq. (1)] and corresponding analysis of variance [Eq. (2)] for data that are indexed by the dihedral group D_4. For simplicity of notation, let $\mathbf{x}'=(u,r,R,\rho,h,d,v,D)$ indicate the vector of experimental data indexed by the dihedral group D_4. Here we write $\mathbf{x}$ as a column vector in $\mathbb{R}^8$ and denote its transpose by $\mathbf{x}'$. The components u, r, ρ and R point to the data obtained following a 0, 90, 180, 270 deg (clockwise) experimental rotation, respectively, and h,d,v,D indicate the resulting data after horizontal, vertical, 45 deg line and 135 deg line experimental axial reflections, respectively. The vector $\mathbf{x}$ constitutes, consequently, an example of structured data. For the moment, $\mathbf{x}$ is a vector in $\mathbb{R}^8$ and a sample of size 1 from the dihedral experiment. The sampling aspects will be discussed subsequently.

The reduction of the data space ($\mathbb{R}^8$) and the determination of their invariants follow from the canonical projections ($\mathcal{P}$) associated with the left regular representation of D_4 (the group acting on itself). Indicating the regular representation by ϕ, the canonical projections are determined by

$$\mathcal{P}_i = \frac{n_i}{|G|}\sum_{\tau\in D_4}\bar{\chi}_i(\tau)\phi(\tau), \qquad j=1,\ldots,5,$$

where $\chi_1,\ldots,\chi_h$ are the distinct irreducible characters of D_4 shown in matrix (6), of dimensions $n_1=n_2=n_3=n_4=1$, $n_5=2$, respectively.[19] The evaluation of these projections give $\mathcal{P}_1=J_8/8$, where J_8 is the 8×8 matrix of ones,

$$\mathcal{P}_2=\frac{1}{8}\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$$\mathcal{P}_3=\frac{1}{8}\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\mathcal{P}_4=\frac{1}{8}\begin{bmatrix} 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{bmatrix},$$

$$\mathcal{P}_5=\frac{1}{2}\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Note that for $i,j=1,\ldots,5$ we have $I=\Sigma_i\mathcal{P}_i$, $\mathcal{P}_i\mathcal{P}_j=0$, $i\neq j$, $\mathcal{P}_i^2=\mathcal{P}_i$. The invariants on the data are the quantities $\mathcal{P}_1\mathbf{x},\ldots,\mathcal{P}_5\mathbf{x}$. These are the summaries of data that need to be retained for analysis and interpretation. Each invariant $\mathcal{P}\mathbf{x}$ reduces according to (isomorphic copies of) the corresponding irreducible representation. The invariants, obtained by direct evaluation of $\mathcal{P}_1\mathbf{x},\ldots,\mathcal{P}_5\mathbf{x}$, are

1. $u+r+R+\rho+h+d+v+D$;
2. $u+r+R+\rho-h-d-v-D$;
3. $u-r+R-\rho+h-d+v-D$;
4. $u-r+R-\rho-h+d-v+D$;
5. $u-R, r-\rho, h-v, d-D$.

Their dimensions, or degrees of freedom (DF), are, respectively, 1, 1, 1, 1, and 4. The resulting decomposition,

$$\|\mathbf{x}\|^2 = \mathbf{x}'\mathcal{P}_1\mathbf{x} + \cdots + \mathbf{x}'\mathcal{P}_5\mathbf{x},$$

of the sum of squares $\mathbf{x}'\mathbf{x}$, relative to the usual Euclidian norm (and underlying geometry), is summarized in matrix (7):

$$\left[\begin{array}{ccc} \text{source} & \mathbf{x}'\mathcal{P}\mathbf{x} & \text{DF} \\ \mathcal{P}_1 & (u+r+R+\rho+h+d+v+D)^2/8 & 1 \\ \mathcal{P}_2 & (u+r+R+\rho-h-d-v-D)^2/8 & 1 \\ \mathcal{P}_3 & (-u+r-R+\rho-h+d-v+D)^2/8 & 1 \\ \mathcal{P}_4 & (-u+r-R+\rho+h-d+v-D)^2/8 & 1 \\ \mathcal{P}_5 & [(u-R)^2+(r-\rho)^2+(h-v)^2+(d-D)^2]/2 & 4 \\ \text{total} & u^2+r^2+R^2+\rho^2+h^2+d^2+v^2+D^2 & 8 \end{array}\right]. \tag{7}$$

The following interpretations are relevant:

1. $\mathbf{x}'\mathcal{P}_1\mathbf{x}$ is the overall reference constant.
2. $\mathbf{x}'\mathcal{P}_2\mathbf{x}$ compares axial and rotational total effects and suggests the parametric hypothesis $\mu_{u+r+R+\rho} = \mu_{h+d+v+D}$ for the expected value (μ) of the corresponding data.
3. $\mathbf{x}'(\mathcal{P}_3+\mathcal{P}_4)\mathbf{x}=[(u+R-r-\rho)^2+(h+v-d-D)^2]/4$, determined within-rotation and within-reflection differences. This component is suitable for testing the joint parametric hypotheses $\mu_{u+R}=\mu_{r+\rho}$ and $\mu_{h+v}=\mu_{d+D}$.
4. $\mathbf{x}'\mathcal{P}_5\mathbf{x}$ is a more specific assessment of within-rotation and within-reflection differences. It can be used to assess the joint parametric hypotheses $\mu_u=\mu_R$, $\mu_r=\mu_\rho$, $\mu_h=\mu_v$, and $\mu_d=\mu_D$.

Other hypotheses can be formulated by reparameterization. For example, if

$$x(\eta^k\tau^l) = kr + la, \qquad k=0,1,2,3, \qquad l=0,1,$$

where r and a represent axial and rotational symmetry effects, respectively, then the components $\mathbf{x}'\mathcal{P}_2\mathbf{x}=2a^2$ and $\mathbf{x}'\mathcal{P}_3\mathbf{x}=2r^2$ clearly separate axial symmetry from rotational symmetry effects.

In every case, however, the statistical assessment of the parametric hypotheses of interest, using the argument of decomposing the sum of squares (or analysis of variance), follows from the well-known theory for the probability distribution of quadratic forms under the assumptions of the Fisher–Cochran theorem.[21] Moreover, when a sample of size N is obtained at each one of the dihedral symmetries, the new sum of squares decomposition $\mathbf{x}'\mathbf{x}$ for the $8\times N$ data points is then obtained from the new canonical decomposition

$$I = I_8 \otimes I_N = \sum_{i=1}^{5} \mathcal{P}_i \otimes (\mathcal{A}+\mathcal{Q}),$$

where $\mathcal{A}=J_N/N$ is the averaging projection in $\mathbb{R}^N$ (J_N is the $N\times N$ matrix of ones) and $\mathcal{Q}=I_N-\mathcal{A}$ is its orthogonal complement in $\mathbb{R}^N$. It is precisely the tensoring of the two decompositions, namely,

$$I_8 = \mathcal{P}_1 + \cdots + \mathcal{P}_5, \qquad I_N = \mathcal{A} + \mathcal{Q},$$

that introduces the error component in the decomposition of $\mathbf{x}'\mathbf{x}$ against which the usual F ratios in the analyses of variance can be obtained.[17] The decomposition of $\mathbf{x}'\mathbf{x}$ extends to decompositions of the form $\mathbf{x}'\Sigma\mathbf{x}$ to accommodate a potential structure of covariance among the components of the structured data vector $\mathbf{x}$.

4. DIHEDRAL FOURIER ANALYSIS: AN EXAMPLE FROM OPHTHALMIC OPTICS

In ophthalmic optics, the simplest representation of any astigmatic (i.e., spherocylindrical) surface curvature corresponds to a surface with the direction of the steep (maximum, κ_s) and flat (minimum, κ_f) curvatures oriented with a 90 deg angular separation.[28] This is simply the Euler theorem of classical differential geometry. The resulting refractive profile,

$$\pi(\theta) = (n-n')[\kappa_s\cos^2(\theta-\alpha) + \kappa_f\sin^2(\theta-\alpha)],$$
$$0 \leqslant \theta \leqslant 2\pi, \quad 0 \leqslant \alpha \leqslant \pi, \tag{8}$$

can be expressed as $\pi(\theta)=s+c\cos^2(\theta-\alpha)$, where $s=(n-n')\kappa_f$, $c=(n-n')(\kappa_s-\kappa_f)$, and α are, respectively, the spherical, cylindrical, and axial (or reference angle for the $\{k_s,k_f\}$ orthogonal directions) components of the spherocylindrical corrective element, and n, n' are refractive indices. The associated refractive power matrix, using the standard notation, is given by

$$F = \begin{bmatrix} s+c\sin^2(\alpha) & -c\sin(\alpha)\cos(\alpha) \\ -c\sin(\alpha)\cos(\alpha) & s+c\cos^2(\alpha) \end{bmatrix}$$
$$= \begin{bmatrix} S-C_+ & -C_x \\ -C_x & S+S_+ \end{bmatrix}. \tag{9}$$

The right-hand side notation is from Campbell,[18,29] in which $S=s+c/2$, $C_+=(c/2)+\cos(2\alpha)$, $C_x=(c/2)\sin(2\alpha)$. We observe that the scalars (s,c,α), respectively, the sphere, cylinder, and axis, form the numerical power matrix F.

To decompose F in the form $\Sigma_{\sigma\in G}x(\sigma)\beta(\sigma)$, with respect to a family $\{\beta(\sigma);\sigma\in G\}$ of elementary operators we make use of the irreducible 2-dim representation of D_4 described in Section 2. The matrices described in Eqs. (4) and (5) are precisely those defining the refractive group.[18] To determine the coefficients $x(\sigma)$ in the decomposition of F relative to these operators we equate the Fourier transforms according to

$$\hat{x}(\eta) = \begin{cases} F & \text{if } \eta=\beta, \\ \operatorname{tr} F & \text{if } \eta\neq\beta, \end{cases}$$

and determine the coefficients $\{x(\tau);\tau\in D_4\}$ using the Fourier-inverse formula[19]

$$x(\tau)=\sum_{\eta}\frac{n_{\eta}}{|G|}\mathrm{tr}[\eta(\tau^{-1})\hat{x}(\eta)], \qquad \tau\in D_4, \tag{10}$$

where the sum is over the irreducible representations (η) of D_4, n_{η} is the dimension of η, and $|G|=8$ is the number of elements in $G=D_4$. That is, we evaluate the inverse-Fourier formula (10) by assigning $\hat{x}(\eta)=F$ when $\eta=\beta$, the two-dimensional irreducible representation described in matrices (4) and (5), and $\hat{\mathbf{x}}(\eta)=\mathrm{tr}\,F$ when η is any one of the other four one-dimensional irreducible representations.

The resulting data are then indexed by D_4. These are the coefficients in the decomposition of F relative to the refractive group. Matrix (11) shows the solution to the Fourier-inverse Eq. (10) for the power matrix F shown in expression (9).

$$\left[\begin{array}{lrr} & \text{Rotational} & \text{Axial} \\ & \text{Coefficients} & \text{Coefficients} \\ j & x(\eta^j) & x(\eta^j\tau) \\ \hline 0 & 3(2s+c)/4 & -c\cos(2\alpha)/4 \\ 1 & 0 & -c\sin(2\alpha)/4 \\ 2 & (2s+c)/4 & c\cos(2\alpha)/4 \\ 3 & 0 & c\sin(2\alpha)/4 \end{array}\right]. \tag{11}$$

It turns out that the coefficients indexing the refractive group, shown in matrix (11), are exactly the coefficients,

$$C_0=c\cos(2\alpha), \qquad C_{45}=c\sin(2\alpha),$$

$$M=[s+(s+c)]/2=s+c/2,$$

appearing in Humphrey's principle of astigmatic decomposition.[30,31] That is, the solutions

$$x(1)=3M/2, \quad x(\eta^2)=M/2, \quad x(\tau)=-C_0,$$

$$x(\eta\tau)=-C_{45}, \quad x(\eta^2\tau)=C_0, \quad x(\eta^3\tau)=C_{45},$$

generated by the dihedral Fourier-inverse method coincide exactly with Humphrey's astigmatic decomposition. The quantity M is easily recognized to be nothing more than the spherical equivalent of the lens. In particular, the statement[28] that "when expressed in such form, cylinders become additive" follows from the additive properties of the vector space $(\mathbb{R}^8)$ defined by the underlying group algebra. It is within this vector space that statistical analysis should then be carried on. A numerical example illustrating the evaluation of the dihedral coefficients is described in Appendix A.

5. CONCLUSIONS AND DISCUSSION

1. In this paper we have introduced the notion of data indexed by the dihedral group D_4. We have shown that these data appear as the coefficients in the Fourier transforms over the irreducible representations of D_4. The refractive group of Campbell[18] coincides with the only 2-dim irreducible representation of D_4 and the data indexed by this group are exactly the coefficients appearing in Humprey's astigmatic decomposition. The correspondence follows directly from the dihedral Fourier-inverse formula. The fact that D_n for n even has $(n/2)-1$ irreducible representations of dimension 2 suggests the study of the correspondence between refractive groups and astigmatic decompositions determined by arbitrarily larger dihedral groups.

2. An immediate application of the methods described in this paper is to the study of the correspondence between geometric optics and the vector algebra from the point of view of classic Fourier analysis as proposed by Thibos *et al.*[32] Refractive errors in the visual optics are specified by a sphere, cylinder, and axis. This formulation is not easily amenable to statistical analysis (say, for example, in a myopia development study). There are two problems with this: (i) the astigmatic component is specified in polar form and (ii) the sphere and cylindrical power are not independent of each other. Thibos proposed a representation by a single power vector in a three-dimensional dioptric space. We have shown in detail the nature of this space and emphasized the fact that optical lens problems/statistics should be studied in this dioptric space. The dihedral Fourier-inverse method proposed in this paper provides a formal and general connection between the original data and the data in a linear subspace of the dioptric space.

3. The arguments described here lead to applications in a large class of data from the psychophysics of human perception of symmetry. For example, the symmetry in the human face is quite accurately perceived by humans.[33] Experimental protocols can be designed to probe the subject's perception of stimuli realized by the action of two-dimensional representations of D_n acting on certain planar images of interest. In these experiments, the data obtained are directly indexed by D_n. The canonical projections and corresponding invariants obtained in Section 3 show which parametric hypotheses can be formally addressed within the context of the Fisher–Cochran theorem. Other applications where the analysis of data indexed by a group of symmetries can be envisioned include those related to anatomic symmetry between fellow eyes,[34] parallel visual processes in symmetry perception in normal vision,[35] and visual signaling by asymmetry[36–38] on hemispherical asymmetry, among others.

4. The group-theoretic methods described in this paper are relevant to the determination of the covariance structure of astigmatic (corneal) curvature surface data[39,40] and in the presence of symmetrically dependent observations.[41–44]

5. Higher-order dihedral Fourier analyses may be obtained by tensoring. For example, in D_4, the tensor representation $\beta\otimes\beta$ is an irreducible representation of the product group $G\times G$ and can be used to decompose a 4×4 linear operator with coefficients in the group algebra of $G\times G$. Similarly, for $n=6$, the four representations $\beta_j\otimes\beta_k$, $j,k=1,2$ determined by the two 2-dim irreducible representations of D_6 are irreducible. These matrices form a new set of elementary 4×4 operators in the expansion of a generic 4×4 operator. These aspects need to be better understood for arbitrary values of n and tensoring factors $\beta\otimes\ldots\otimes\beta$.

6. The dihedral decomposition of basic optical instruments can be obtained from the representation described

in Section 4. These include compensators, rotators, and polarizers. The dihedral decomposition of coherency matrices also follows from the same principles introduced in the present paper. These decompositions will appear in a future companion paper.

APPENDIX A: NUMERICAL EXAMPLE

Figure 1 illustrates the power profiles (in polar coordinates) for $s=4.25$, $c=-1.5$, $\alpha=20$ deg and $s=-2.75$, $c=1.00$, $\alpha=10$ deg. We will map the data from the corresponding power operators

$$(s,c,\alpha)=(4.25,-1.5,20\text{ deg})\rightarrow F_1=\begin{bmatrix}4.0745 & 0.48207\\ 0.48207 & 2.9255\end{bmatrix},$$

$$(s,c,\alpha)=(-2.75,1.0,10\text{ deg})\rightarrow F_2$$
$$=\begin{bmatrix}-2.7198 & -0.17101\\ -0.17101 & -1.7802\end{bmatrix}$$

into the (refractive) group algebra using the dihedral Fourier-inverse formula (10). The solutions are given, respectively, by

$$F_1{:}\frac{1}{4}\left[\begin{array}{c|cc} j & x(\eta^j) & x(\eta^j\tau)\\ \hline 0 & 21.0 & 1.1491\\ 1 & 0 & 0.96414\\ 2 & 7.0 & -1.1491\\ 3 & 0 & -0.96414\end{array}\right],$$

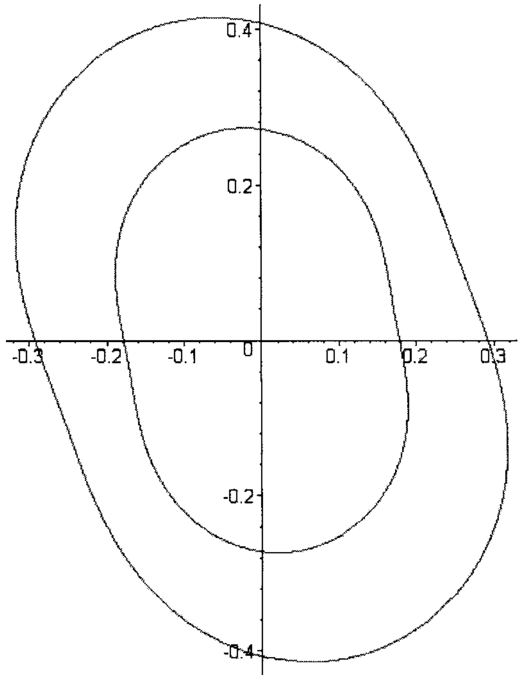

Fig. 1. Refractive profile for s=4.25, c=−1.5, α=20 deg (outer contour) and s=−2.75, c=1.00, α=10 deg (inner contour).

$$F_2{:}\frac{1}{4}\left[\begin{array}{c|cc} j & x(\eta^j) & x(\eta^j\tau)\\ \hline 0 & -13.500 & -0.93970\\ 1 & 0 & -0.34202\\ 2 & -4.5000 & 0.93970\\ 3 & 0 & 0.34202\end{array}\right].$$

These are the coefficients for the elementary operators described in matrices (4) and (5) in the decomposition of the power operators. The operator F_1 decomposes as

$$F_1=\frac{1}{4}[21\beta(1)-7\beta(\eta^2)+1.1491\beta(\tau)+0.69414\beta(\eta\tau)$$
$$-1.1491\beta(\eta^2\tau)-0.69414\beta(\eta^3\tau)],$$

whereas

$$F_2=\frac{1}{4}[-13.5\beta(1)-4.5\beta(\eta^2)-0.9397\beta(\tau)-0.342\beta(\eta\tau)$$
$$+0.9397\beta(\eta^2\tau)+0.342\beta(\eta^3\tau)],$$

with

$$\beta(1)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\qquad \beta(\eta^2)=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},$$

$$\beta(\tau)=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\qquad \beta(\eta\tau)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},$$

$$\beta(\eta^2\tau)=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\qquad \beta(\eta^3\tau)=\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}.$$

These linear combinations are the elements defining the (refractive) group algebra. They have a vector space structure, allowing for the addition and scalar multiplication of the (vector of) coefficients obtained above, and a multiplicative structure induced by the group multiplication. Note the correspondence with the notation in Campbell[18]: $\beta(1)=-\beta(\eta^2)=S$ is the spherical component, $\beta(\tau)=-\beta(\eta^2\tau)=C_+$ is one of the cross-cylinder components, and $\beta(\eta\tau)=-\beta(\eta^3\tau)=C_x$ is the other.

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