

The following example illustrates the role of different group actions on the reduction of data indexed by the symmetries (S_3) of a regular triangle. We remark that here the structure of interest is in itself a group.

In the first part of this example we consider the action

$$\left[\begin{array}{c|cccccc} \varphi(\tau, \sigma) = \tau\sigma\tau^{-1} & a & b & c & d & e & f \\ \hline a = 1 & a & b & c & d & e & f \\ b = (12) & a & b & d & c & f & e \\ c = (13) & a & d & c & b & f & e \\ d = (23) & a & c & b & d & f & e \\ e = (123) & a & d & b & c & e & f \\ f = (132) & a & c & d & b & e & f \end{array} \right]$$

of S_3 on itself by conjugacy, so that the resulting orbits are exactly the conjugacy classes of S_3 . The canonical decompositions associated with this action are given by

$$\mathcal{P}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 0 & 0 & 1/2 & 1/2 \end{bmatrix}, \quad \mathcal{P}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & -1/2 \\ 0 & 0 & 0 & 0 & -1/2 & 1/2 \end{bmatrix},$$

and

$$\mathcal{P}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2/3 & -1/3 & -1/3 & 0 & 0 \\ 0 & -1/3 & 2/3 & -1/3 & 0 & 0 \\ 0 & -1/3 & -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

of dimensions 3, 1 and 2, respectively. The invariants $\mathcal{P}\mathbf{x}$ in the data vector

$$\mathbf{x}' = (a, b, c, d, e, f),$$

indicated by their labels for simpler notation, associated with the $\mathcal{P}_1, \mathcal{P}_2$ and \mathcal{P}_3 are, respectively,

$$\left\{ a, \frac{1}{3}(b + c + d), \frac{1}{2}(e + f), \{\pm(e - f)\} \right\},$$

and

$$\left\{ \frac{1}{3}\left[b - \frac{c + d}{2}\right], \frac{1}{3}\left[c - \frac{b + d}{2}\right], \frac{1}{3}\left[d - \frac{b + c}{2}\right] \right\}.$$

The components of the decomposition of $\mathbf{x}'\mathbf{x}$ are then

$$\mathbf{x}'\mathcal{P}_1\mathbf{x} = a^2 + \frac{1}{3}(a + b + c)^2 + \frac{1}{2}(e + f)^2, \quad \mathbf{x}'\mathcal{P}_2\mathbf{x} = \frac{1}{2}(e - f)^2,$$

and

$$\mathbf{x}'\mathcal{P}_3\mathbf{x} = \frac{2}{3}(b^2 + c^2 + d^2 - bc - bd - cd).$$

The parametric hypotheses afforded by this reduction are

- (1) H : $e = f$ that the two (non-trivial) rotation parameters are the same;
- (2) H : $b = c = d$, that all transposition parameters are the same.

In the second part of this symmetry study we consider the regular action $\varphi(\tau, \sigma) = \sigma\tau$ of S_3 on itself. This action is generated by the Cayley table of S_3 . In contrast, the resulting canonical reductions now are

$$\mathcal{P}_1 = 1/6 \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad \mathcal{P}_2 = 1/6 \begin{bmatrix} 1 & -1 & -1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix}$$

and

$$\mathcal{P}_3 = 1/3 \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Their dimensions are 1, 1 and 4, respectively. Under the regular action, $x'x$ now decomposes as

$$x'\mathcal{P}_1x = \frac{1}{6}(a + b + c + d + e + f)^2, \quad x'\mathcal{P}_2x = \frac{1}{6}(a + e + f - b - c - d)^2,$$

and

$$x'\mathcal{P}_3x = \frac{1}{3}((2a - e - f)a + (2b - c - d)b + (-b + 2c - d)c \\ + (-b - c + 2d)d + (-a + 2e - f)e + (-a - e + 2f)f).$$

The parametric hypotheses afforded by this reduction are

- (1) H : $a + e + f = b + c + d$ that the parameter sum of rotations equals the parameter sum of transpositions;
- (2) H : $a = e = f$ and $b = c = d$, of homogeneity of rotation parameters and of homogeneity of transposition parameters.

Observe that these hypotheses are disjoint and that the corresponding quadratic forms are null when each one of them obtains.