SYMMETRY STUDIES - A BRIEF OVERVIEW
MARLOS VIANA

1. Symmetry and measurement

George Pólya, in his introduction to mathematics and plausible reasoning, observes that *A great part of the naturalist's work is aimed at describing and classifying the objects that he observes. A good classification is important because it reduces the observable variety to relative few clearly characterized and well ordered types.*

Pólya's narrative introduces us directly to the practical aspect of partitioning a large number of objects by exploring certain rules of equivalence among these objects. This is how symmetry will be understood in the present text: as a set of rules with which we may describe certain commonalities or invariants among objects or concepts. The classification of crystals, for example, is based on their symmetries.

Included in the naturalist's methods of description is the delicate notion of measuring something on these objects and recording their data, so that the classification of the objects may be related to the classification or partitioning of their corresponding data. Pólya's picture also includes the notion of interpreting, or characterizing, the resulting types of varieties. That is, the naturalist has a better result when he can explain why certain varieties fall into the same type or category. The interplay between symmetry and data is the driving theme underlying any symmetry study.

In practice, a symmetry study starts with the identification of data sets in which symmetry transformations can be identified. That is, the identification of a *structured data*.

Here is an example of a simple structure from molecular biology. A biological sequence is a finite string of symbols from a finite alphabet ($\mathcal{A}$) of residues, such as the linear string

`cttgggatattgatgatctgtagtgctacagaaaaattgtgggtcacagtct`,

in which the symbols are letters in the alphabet $\mathcal{A} = \{\mathrm{a, g, t, c}\}$. Here the symbols represent molecules in DNA sequences. We view the set

$$\mathrm{V} = \{\mathrm{ttt,\ ttc,\ tta}, \ldots, \mathrm{gga,\ ggg}\}$$

defined by these 64 short sequences (s) in length of three written with a four-letter alphabet $\mathcal{A}$, as a structure indexing potential molecular constructs or measurements, x(s), such as the triplet's molecular weight, or the frequencies with which these short sequences appear in the reference, longer sequence shown above. In that sense, then,

$$\mathrm{x(ttt),\ x(ttc),\ x(tta)}, \ \ldots, \mathrm{x(gga),\ x(ggg)}$$

are data indexed by the structure V, or, shortly, a *structured data*.

2. Connecting algebraic and statistical tools: making it work

The analysis of structured data explores the symmetries in the structure (V), with the purpose of simplifying V to better explain the data x. The points $\mathrm{s} \in \mathrm{V}$ are labels making possible the identification of potential events, where annotations or scalar experimental realizations $\mathrm{x(s)} \in \mathcal{V}$ are observed and recorded. Typically, $\mathcal{V}$ is an affine subspace of a real or complex vector space. Experimental questions dictate the symmetries of interest- typically those defined by a finite group (G) of transformations (τ). In the molecular biology structure illustrated above, there are symmetry transformations (or permutations) that can be defined for

 This overview is partially abstracted from the forthcoming title *Structured Data- An Introduction to the Study of Symmetry in Applications* by **Cambridge University Press**- Comments to viana@uic.edu- M. Viana, University of Illinois, Chicago, USA. Revised July 13, 2005.

the letter positions $\{1, 2, 3\}$ in each short sequence and there are symmetries that can be defined in the alphabet $\mathcal{A} = \{a, g, c, t\}$.

These symmetries when applied to the labels in V, according to a definition rule φ (technically a group action), simplify or factor these labels into disjoint orbits so that $V = \mathcal{O}_1 \cup \ldots \cup \mathcal{O}_m$, $m \geq 1$. This factorization of V, in turn, appears as linear transformations $\rho(\tau)$ in the data vector space $\mathcal{V}$, defined by the changing of the canonical basis $\{e_s; s \in V\}$ of $\mathcal{V}$ into the basis $\{e_{\varphi(\tau,s)}; s \in V\}$, for each $\tau \in G$. The resulting factorization $\mathcal{V} = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_h$ in the data vector space is the consequence of defining a set of h orthogonal projections ($\mathcal{P}$), which are linear combinations of these permutation matrices with real or complex scalar coefficients (technically, the characters of the irreducible representations of G). There are as many projections as the number of irreducible characters in the group under consideration.

In particular, then, the identity operator I in $\mathbb{R}^v$ decomposes as $I = \mathcal{P}_1 + \mathcal{P}_2 + \ldots + \mathcal{P}_h$. Moreover, $\mathcal{P}_i\mathcal{P}_j = 0$ for $\imath \neq j$ and $\mathcal{P}_i^2 = \mathcal{P}_i$, $i = 1, \ldots, h$ and the decomposition is said to be a *canonical decomposition*. It then follows that the basic partition

$$||x||^2 = (x|x) = (x|\mathcal{P}_1 x) + (x|\mathcal{P}_2 x) + \ldots + (x|\mathcal{P}_h x)$$

of the sum of squares for a particular inner product $(\cdot|\cdot)$ of interest (e.g., Euclidean, Hermitian, symplectic) can be obtained. For normally distributed data, the Fisher-Cochran theorem for the probability distribution of quadratic forms leads to varied forms of analyses of variance, within which parametric hypotheses may be defined and interpreted. In addition, when the canonical decomposition is applied to the regular representation (the group acting on itself), a spectral analysis for the structured data is then obtained. This includes the Fourier and Fourier-Inverse analyses over finite groups.

These are the elemntary algebraic and statistical arguments in the analysis of structured data introduced in *Symmetry Studies*, Viana (2003), which are consequence of the theory of canonical decompositions of a representation, e.g., Serre's (1977) text on Linear Representations of Finite Groups. See also Viana (2005).

3. Specifications of a symmetry study

Typical parameters of a symmetry study include:

- A structure, or a set of labels V. Data $x, y, \ldots$ are indexed by V, with $|V| = v$ points denoted generically, by s, so that the data have the form $x = (x(s))_{s \in V}$. These are ordered vectors in $\mathbb{R}^v$ up to an arbitrary ordering of the points in the structure V - Shortly, the data $x, y, \ldots$ are termed structured data;
- Symmetries are those of a finite group, G, of order g, acting on V;
- Group actions $\varphi : G \times V \to V$ of G on V;

Typical specifications:

1. Data structures:
   (a) The set $V = C^L$ of all mappings $s : L \to C$, with $L = \{1, \ldots, \ell\}$ and $C = \{1, \ldots, c\}$. Here ℓ and c are also parameters which will depend on applications to be described subsequently;
   (b) The set product $V = L \times C$ and its symmetrically stable subsets, with L and C as described above and in particular the space $V = L \times \Omega$ of circularly annotated data, with $L = \{1, \ldots, \ell\}$ and $\Omega = \{\omega; \omega^c = 1\}$. Here, as well, ℓ and c are parameters dependent on applications to be described subsequently;
   (c) Dual structures $V \otimes \ldots \otimes V$.
2. Symmetries:
   (a) $G = S_n$, the group of permutations of $\{1, \ldots, n\}$;
   (b) $G = C_n$, the cyclic subgroup of S_n;
   (c) $G = D_n$, the (dihedral) group of rotational and axial symmetries of the regular n-sided polygon;
3. Geometries: Euclidean under the usual inner product;
4. Group actions $\varphi : G \times V \to V$:

(a) G acting on a generic structure V. Left action, right action, action by conjugation (permutation actions);
(b) G acting on itself (V = G, regular actions).

4. The canonical projections

Theorem 1 (Canonical Decomposition)**.** Let ρ be a linear representation of G into $\mathrm{GL}(\mathcal{V})$, $\rho_1, \ldots, \rho_h$ the distinct irreducible representations of G, with corresponding characters $\chi_1, \ldots, \chi_h$ and dimensions $n_1, \ldots, n_h$. Then,

$$\mathcal{P}_i = \frac{n_i}{g} \sum_{\tau \in G} \overline{\chi}_i(\tau)\rho(\tau),$$

is a projection of $\mathcal{V}$ onto a subspace $\mathcal{V}_i$, sum of m_i isomorphic copies of the irreducible subspaces associated with ρ_i, $i = 1, \ldots, h$. Moreover, $\mathcal{P}_i\mathcal{P}_j = 0$, for $i \neq j$, $\mathcal{P}_i^2 = \mathcal{P}_i$ and $\sum_i \mathcal{P}_i = I_v$, where $v = \dim \mathcal{V} = \sum_{i=1}^h m_i n_i$.

Example 1. Let V indicate the set $V = \{uu, yy, uy, yu\}$ of binary sequences in length of two, equivalently, the set of all mappings s from $L = \{1, 2\}$ into L. Let $G = S_2$. Consider the action $\varphi(\tau, s) = s\tau^{-1}$ which classifies the sequences by symmetries in the position of the residues:

$$\varphi : \left[\begin{array}{c|cccc} \tau\backslash s & uu & yy & uy & yu \\ \hline 1 & uu & yy & uy & yu \\ t = (12) & uu & yy & yu & uy \end{array}\right].$$

The permutation representation of S_2 defined by the left action is then

$$\rho(1) = I_4, \quad \rho(t) = \begin{bmatrix} \boxed{1} & & \\ & \boxed{1} & \\ & & \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} \end{bmatrix}.$$

The irreducible characters are the characters χ_1 and χ_2 of the trivial and the signature representations, respectively:

$$\left[\begin{array}{c|cc} \tau & 1 & t \\ \hline \chi_1 & 1 & 1 \\ \chi_2 & 1 & -1 \end{array}\right].$$

These representations have dimension equal to 1, so that in Theorem 1 $n_1 = n_2 = 1$ and $g = 2$. Therefore

$$\mathcal{P}_1 = \frac{1}{2}[\overline{\chi}_1(1)\rho(1) + \overline{\chi}_1(t)\rho(t)] = \frac{1}{2}[\rho(1) + \rho(t)] = \begin{bmatrix} \boxed{1} & & \\ & \boxed{1} & \\ & & \boxed{\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}} \end{bmatrix},$$

$$\mathcal{P}_2 = \frac{1}{2}[\overline{\chi}_2(1)\rho(1) + \overline{\chi}_2(t)\rho(t)] = \frac{1}{2}[\rho(1) - \rho(t)] = \begin{bmatrix} \boxed{0} & & \\ & \boxed{0} & \\ & & \boxed{\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}} \end{bmatrix}.$$

We have $I_4 = \mathcal{P}_1 + \mathcal{P}_2$, $\mathcal{P}_1\mathcal{P}_2 = 0$, $\mathcal{P}_1^2 = \mathcal{P}_1$, $\mathcal{P}_2^2 = \mathcal{P}_2$. Note that, correspondingly, $\mathcal{V} = \mathbb{R}^4$ decomposes into the sum $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$ of stable subspaces, of dimensions 1, 1, 2, respectively. $\square$

5. Spectral analysis

When the data are indexed by a group G we look at actions of G on itself (the V = G case) to simplify the structured data. Group algebras and Fourier Analysis are then of natural interest. The following theorem relates the canonical decomposition (Theorem 1) with the spectral analysis of data indexed by G.

Theorem 2. If x is a data vector indexed by the finite group G and $\widehat{x}(\beta) = \sum_{\tau \in G} x(\tau)\beta(\tau)$ is its Fourier transform at the irreducible representation β then, conversely,

$$x(\tau) = \sum_{\beta} \frac{n_\beta}{g} \text{tr}\,[\beta(\tau^{-1})\widehat{x}(\beta)],$$

where the sum is over all irreducible representations of G.

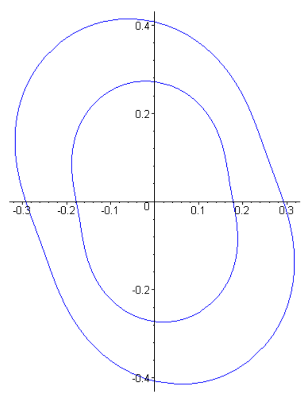
6. Refractive power decomposition: An example from ophthalmic optics

In ophthalmic optics, e.g., Bennett and Rabbetts (1984), the simplest representation of any astigmatic (i.e., sphero-cylindrical) corneal surface curvature corresponds to a surface with the direction of the steep (maximum, κ_s) and flat (minimum, κ_f) curvatures oriented with a 90 deg angular separation. This is simply Euler Theorem of classical differential geometry. The resulting refractive profile,

$$\pi(\theta) = (\eta - \eta')[\kappa_s \cos^2(\theta - \alpha) + \kappa_f \sin^2(\theta - \alpha)], \quad 0 \le \theta \le 2\pi, \quad 0 \le \alpha \le \pi, \tag{6.1}$$

can be expressed as $\pi(\theta) = s + c\cos^2(\theta - \alpha)$, where $s = (\eta - \eta')\kappa_f$, $c = (\eta - \eta')(\kappa_s - \kappa_f)$ and α are respectively the spherical, cylindrical and axial (or reference angle for the $\{k_s, k_f\}$ orthogonal directions) components of the spherocylindrical corrective element, and η', η are refractive indices. Figure 6.1 illustrates the power profiles (in polar coordinates) for $s = 4.25, c = -1.5, \alpha = 20$ deg and $s = -2.75, c = 1.00, \alpha = 10$ deg. The

Figure 6.1. Refractive profile for $s = 4.25, c = -1.5, \alpha = 20$ deg (outer contour) and $s = -2.75, c = 1.00, \alpha = 10$ deg (inner contour).



associated refractive power matrix, using the standard notation (Long, 1976), is given by

$$F = \begin{bmatrix} s + c\sin^2(\alpha) & -c\sin(\alpha)\cos(\alpha) \\ -c\sin(\alpha)\cos(\alpha) & s + c\cos^2(\alpha) \end{bmatrix} = \begin{bmatrix} S - C_+ & -C_x \\ -C_x & S + S_+ \end{bmatrix}. \tag{6.2}$$

The RHS notation is from C. Campbell, e.g., Campbell (1997), Campbell (1994), in which $S = s + c/2$, $C_+ = (c/2) + \cos(2\alpha)$, $C_x = (c/2)\sin(2\alpha)$.

We observe that the scalars (s, c, α), respectively the sphere, cylinder and axis, form the numerical power matrix F, e.g.,

$$(s, c, \alpha) = (4.25, -1.5, 20\,\text{deg}) \to F = \begin{bmatrix} 4.0745 & 0.48207 \\ 0.48207 & 2.9255 \end{bmatrix}.$$

The first step is equating the data matrix F with an irreducible two-dimensional representation. One choice, in the case of geometric optics, is the two-dimensional irreducible representation, β, of the dihedral group D_4. That is, we set the Fourier equation

$$F = \widehat{x}(\beta),$$

and solve it for the data $\{x(\tau); \tau \in D_4\}$ using the formula

$$x(\tau) = \sum_{\rho} \frac{\rho(1)}{g} \text{tr } [\rho(\tau^{-1})F], \quad \tau \in D_4,$$

where the sum is over the irreducible representations of D_4. That is, we evaluate the inverse-Fourier formula $x(\tau) = \sum_{\rho} \rho(1)\text{tr } [\rho(\tau^{-1})\widehat{x}(\rho)]/g$ by assigning $\widehat{x}(\rho) = F$ when $\rho = \beta$, the two-dimensional irreducible frequency, and $\widehat{x}(\rho) = \text{tr } F$ when ρ is any one of the four one-dimensional irreducible frequencies.

The resulting data are then indexed by D_4. These are the coefficients in the decomposition of F relative to the refractive group. Matrix 6.3 shows the solution to the Fourier-inverse equation $F = \widehat{x}(\beta)$ in D_4 for the power matrix F shown in expression 6.2. Note that tr F is the contribution of the (four) one-dimensional frequencies to those entries.

(6.3)
$$\frac{1}{4}\left[\begin{array}{c|c|c} j & \text{rotational coefficients } x(\eta^j) & \text{axial coefficients } x(\eta^j\tau) \\ \hline 0 & 3\text{tr F} & -c\cos(2\alpha) \\ 1 & 0 & -c\sin(2\alpha) \\ 2 & \text{tr F} & c\cos(2\alpha) \\ 3 & 0 & c\sin(2\alpha) \end{array}\right], \quad \text{tr F} = 2s + c.$$

It turns out that the coefficients indexing the refractive group, shown in Matrix 6.3, are exactly the coefficients

$$C_0 = c\cos(2\alpha), \;\; C_{45} = c\sin(2\alpha), \;\; M = [s + (s + c)]/2 = s + c/2,$$

appearing in W.E. Humphrey's principle of *astigmatic decomposition* (Humphrey (1976), see also Saunders (1985)). That is, the solutions

$$x(1) = x(\eta^2) = 6M, \;\; x(\tau) = -C_0, \;\; x(\eta\tau) = -C_{45}, \;\; x(\eta^2\tau) = C_0, \;\; x(\eta^3\tau) = C_{45},$$

generated by the dihedral Fourier-inverse method coincide exactly with Humphrey's astigmatic decomposition. The quantity M is easily recognized to be nothing more than the spherical equivalent of the lens. In particular, the statement that *when expressed in such form, cylinders become additive*, in Bennett and Rabbetts (1984), follows from the additive properties of the vector space ($\mathbb{R}^8$) defined by the underlying group algebra. It is within this vector space that statistical analysis should then be carried on. For details, see Lakshminarayanan and Viana (2005).

7. Co-invariants

Elie Cartan, in his 1937 seminal book on the theory of spinors[1] makes the following remark (p.22), adapted to our current notation: *Let two vectors* x *and* y *be referred to the same Cartesian frame of reference and let us consider the* n^2 *products* x_iy_i*; as a result of a rotation they obviously undergo a linear transformation* T*, which also possesses the property that if* T *and* T′ *correspond to the rotations* R *and* R′*, the transformation* TT′ *corresponds to* RR′*. The* n^2 *quantities* x_iy_j *therefore provide a new linear representation of the group of rotations, completely distinct from the two previous ones.*

Proposition 1. If ρ is a linear representation of a finite group G on $GL(\mathbb{R}^v)$ then $\rho \otimes \rho$ is a representation of G on $GL(\mathbb{R}^{v\times v})$ leaving $\{xy'; x, y \in \mathbb{R}^v\}$ invariant.

Example 2. [Co-invariants of bilateral symmetries] Consider two scalar measurements, x and y, obtained on two sides, denoted by 1 and 2, of an experimental unit and let $S_2 = \{1, \tau = (12)\}$ act on $\{1, 2\}$ (bilateral symmetry). We observe the data

$$x' = (x_1, x_2), \quad y' = (y_1, y_2).$$

[1]The English reprint of the original text was published in (1966) by MIT Press. Spinors are interpreted, in Cartan's work, as polarized zero-norm (or isotropic) vectors.

Direct evaluation of $\mathrm{r} = \rho \otimes \rho$ shows that

$$\mathrm{r}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{r}_\tau = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

so that the canonical projections $\mathcal{P}_1 = \frac{1}{2}[\mathrm{r}_1 + \mathrm{r}_\tau]$ and $\mathcal{P}_2 = \frac{1}{2}[\mathrm{r}_1 - \mathrm{r}_\tau]$ on $\mathbb{R}^{2\times 2}$ are

$$\mathcal{P}_1 = 1/2 \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathcal{P}_2 = 1/2 \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix},$$

of dimensions $2, 2$ respectively. The resulting invariant decomposition of $\mathrm{x}'\mathrm{y}$ is

$$\mathrm{xy}' = 1/2 \begin{bmatrix} \mathrm{x}_1\mathrm{y}_1 + \mathrm{x}_2\mathrm{y}_2 & \mathrm{x}_1\mathrm{y}_2 + \mathrm{x}_2\mathrm{y}_1 \\ \mathrm{x}_1\mathrm{y}_2 + \mathrm{x}_2\mathrm{y}_1 & \mathrm{x}_1\mathrm{y}_1 + \mathrm{x}_2\mathrm{y}_2 \end{bmatrix} + 1/2 \begin{bmatrix} \mathrm{x}_1\mathrm{y}_1 - \mathrm{x}_2\mathrm{y}_2 & \mathrm{x}_1\mathrm{y}_2 - \mathrm{x}_2\mathrm{y}_1 \\ -\mathrm{x}_1\mathrm{y}_2 + \mathrm{x}_2\mathrm{y}_1 & -\mathrm{x}_1\mathrm{y}_1 + \mathrm{x}_2\mathrm{y}_2 \end{bmatrix},$$

and in particular,

$$\mathrm{xx}' = 1/2 \begin{bmatrix} \mathrm{x}_1^2 + \mathrm{x}_2^2 & 2\mathrm{x}_1\mathrm{x}_2 \\ 2\mathrm{x}_1\mathrm{x}_2 & \mathrm{x}_1^2 + \mathrm{x}_2^2 \end{bmatrix} + 1/2 \begin{bmatrix} \mathrm{x}_1^2 - \mathrm{x}_2^2 & 0 \\ 0 & -\mathrm{x}_1^2 + \mathrm{x}_2^2 \end{bmatrix}. \tag{7.1}$$

The two components in the canonical decomposition represent, respectively, the co-invariants of intraclass covariance and bilateral variance differentiation. To see this in the usual statistical formulation, we apply the decomposition 7.1 to

$$\mathrm{A} = \frac{1}{\mathrm{n}} \sum_\alpha \mathrm{z}_\alpha \mathrm{z}'_\alpha, \quad \alpha = 1, \ldots, \mathrm{N}$$

where $\mathrm{z}_\alpha = \mathrm{x}_\alpha - \overline{\mathrm{x}}$, $\overline{\mathrm{x}} = \sum_\alpha \mathrm{x}_\alpha/\mathrm{N}$ and $\mathrm{n} = \mathrm{N} - 1$, so that then the canonical decomposition

$$\mathrm{A} = \frac{1}{2}\mathrm{A}_{\text{intraclass}} + \frac{1}{2}(\mathrm{s}_1^2 - \mathrm{s}_2^2) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

obtains. Clearly, it also says that matrix A is an intraclass matrix if and only if the second component in the decomposition vanishes, that it, when $\mathrm{s}_1^2 = \mathrm{s}_2^2$, as well-known. In that case, in fact, A is the usual maximum likelihood estimate of the underlying covariance structure. Algebraically, the first component in the decomposition is a matrix which commutes with all the elements in the permutation representation of S_2. We say that it *intertwines* with the permutation representation. We also way that it has the symmetry of (the permutation representation of) S_2. □

Example 3. Co-invariants for the cyclic group C_4 acting on $\{1, 2, 3, 4\}$. The representation acting on (the ij-indices of) the entries of

$$\mathrm{x} \otimes \mathrm{y} = \mathrm{xy}' = \begin{bmatrix} \mathrm{x}_1\mathrm{y}_1 & \mathrm{x}_1\mathrm{y}_2 & \mathrm{x}_1\mathrm{y}_3 & \mathrm{x}_1\mathrm{y}_4 \\ \mathrm{x}_2\mathrm{y}_1 & \mathrm{x}_2\mathrm{y}_2 & \mathrm{x}_2\mathrm{y}_3 & \mathrm{x}_2\mathrm{y}_4 \\ \mathrm{x}_3\mathrm{y}_1 & \mathrm{x}_3\mathrm{y}_2 & \mathrm{x}_3\mathrm{y}_3 & \mathrm{x}_3\mathrm{y}_4 \\ \mathrm{x}_4\mathrm{y}_1 & \mathrm{x}_4\mathrm{y}_2 & \mathrm{x}_4\mathrm{y}_3 & \mathrm{x}_4\mathrm{y}_4 \end{bmatrix}$$

is $\mathrm{r}_\mathrm{k} = \rho_\mathrm{k} \otimes \rho_\mathrm{k}$, $\mathrm{k} = 0, 1, 2, 3$, where ρ_k is the permutation representation of C_4. Given the character table

k	0	1	2	3
χ_0	1	1	1	1
χ_1	1	i	−1	−i
χ_2	1	−1	1	−1
χ_3	1	−i	−1	i

for C_4, the resulting canonical projections $\mathcal{P}_j = \frac{1}{4}\sum_{k=0}^{3} \overline{\chi_j}(\tau^k)r(\tau^k), \quad j = 1, 2, 3, 4$ each of dimension 4, lead to a decomposition of xy'

$$xy' = \frac{1}{4}\begin{bmatrix} \alpha & \beta & \gamma & \delta \\ \delta & \alpha & \beta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \beta & \gamma & \delta & \alpha \end{bmatrix} + \frac{1}{4}\begin{bmatrix} A & B & C & D \\ -D & -A & -B & -C \\ C & D & A & B \\ -B & -C & -D & -A \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a & c & e & f \\ g & b & d & h \\ -e & -f & -a & -c \\ -d & -h & -g & -b \end{bmatrix},$$

obtained, respectively, from $\mathcal{P}_1$, $\mathcal{P}_3$ and $\mathcal{P}_{24} = \mathcal{P}_2 + \mathcal{P}_4$. We remark that $\mathcal{P}_2$ and $\mathcal{P}_4$ are complex conjugate and that $\mathcal{P}_2^*\mathcal{P}_4 = \mathcal{P}_4^*\mathcal{P}_2 = 0$. We note that the first component in the above decomposition intertwines with C_4, whereas the two other components exhibit a pattern that nearly has the symmetry of C_4. That interpretation is similar to that made for Example 2. The corresponding 16 co-invariants for xy' and 10 co-invariants for xx' are shown in Matrix 7.2. Note that in the decomposition of xx' we have the additional 6 constraints $\beta = \delta$, $B = -D$, $e = h = 0$, $c = g$ and $f = -d$, thus bringing the total number of co-invariants to be $16 - 6 = 10$, the dimension of xx'.

(7.2)

entry	Co-invariants of xy'	Co-invariants of xx'
α	$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$	${x_1}^2 + {x_2}^2 + {x_3}^2 + {x_4}^2$
β	$x_1y_2 + x_2y_3 + x_3y_4 + x_4y_1$	$x_4x_1 + x_2x_1 + x_3x_2 + x_4x_3$
γ	$x_1y_3 + x_2y_4 + x_3y_1 + x_4y_2$	$2\,x_3x_1 + 2\,x_4x_2$
δ	$x_1y_4 + x_2y_1 + x_3y_2 + x_4y_3$	$x_4x_1 + x_2x_1 + x_3x_2 + x_4x_3$
A	$x_1y_1 - x_2y_2 + x_3y_3 - x_4y_4$	${x_1}^2 - {x_2}^2 + {x_3}^2 - {x_4}^2$
B	$x_1y_2 - x_2y_3 + x_3y_4 - x_4y_1$	$-x_4x_1 + x_2x_1 - x_3x_2 + x_4x_3$
C	$x_1y_3 - x_2y_4 + x_3y_1 - x_4y_2$	$2\,x_3x_1 - 2\,x_4x_2$
D	$x_1y_4 - x_2y_1 + x_3y_2 - x_4y_3$	$x_4x_1 - x_2x_1 + x_3x_2 - x_4x_3$
a	$x_1y_1 - x_3y_3$	${x_1}^2 - {x_3}^2$
c	$x_1y_2 - x_3y_4$	$x_2x_1 - x_4x_3$
e	$x_1y_3 - x_3y_1$	0
f	$x_1y_4 - x_3y_2$	$-x_3x_2 + x_4x_1$
g	$x_2y_1 - x_4y_3$	$x_2x_1 - x_4x_3$
b	$x_2y_2 - x_4y_4$	${x_2}^2 - {x_4}^2$
d	$x_2y_3 - x_4y_1$	$x_3x_2 - x_4x_1$
h	$x_2y_4 - x_4y_2$	0

.

□

References

Bennett, A. G. and Rabbetts, R. B. (1984), *Clinical Visual Optics*, Butterworth-Heinermann, Oxford, UK.

Campbell, C. (1994), 'Ray vector fields', *Journal of the Optical Society of America* **11**, 618–622.

Campbell, C. (1997), 'The refractive group', *Optometry and Vision Science* **74**, 381–387.

Cartan, E. (1966), *The Theory of Spinors*, MIT Press, Cambridge, MA.

Humphrey, W. E. (1976), 'A remote subjective refractor employing continuously variable sphere-cylinder corrections', *Opt. Engineering* **15**, 286–291.

Lakshminarayanan, V. and Viana, M. (2005), 'Dihedral representations and statistical geometric optics I: Spherocylindrical lenses', *J. Optical Society of America A - to appear.*

Saunders, H. (1985), 'The algebra of sphero-cylinders', *Ophthal. Physiol. Optics* **5**, 157–163.

Serre, J.-P. (1977), *Linear Representations of Finite Groups*, Springer-Verlag, New York.

Viana, M. (2003), *Symmetry Studies- An Introduction*, IMPA Institute for Pure and Applied Mathematcis Press, Rio de Janeiro, Brazil.

Viana, M. (2005), *Lecture Notes on Symmetry Studies*, EURANDOM and Euler Institute for Discrete Mathematics and its Applications (EIDMA), Eindhoven University of Technology, Eindhoven, NL.