STAT381 **Lia Liu**

Chap 2: Probability Models

List of topics: Section 2.1

1. **Experiment: a process of observations or measurements.**
2. **Outcomes: the results obtained from an experiment.**
3. **Sample space(universal set): the set of all possible outcomes of an experiment.**

   Each outcome in a sample space is called an ELEMENT or a SAMPLE POINT.
4. **Event(subset): An EVENT is a subset of a sample space.**
5. **Two events A and B are called MUTUALLY EXCLUSIVE if $A \bigcap B = \Phi$.**
6. **Discrete sample space**
7. **Continuous sample space**
8. **Def: Probability is a function which maps events of a sample space S into $[0, 1]$**

   P1: $P(A) \geq 0$, **for any event A in S;**

   P2: $P(S) = 1$;

   P3: If $A_1, A_2, ...$ is a sequence of mutually exclusive events of S, then $P(A_1 \bigcup A_2 \bigcup ...) = P(A_1) + P(A_2) + ...$
9. **Complement:** $P(A) = 1 - P(A')$
10. **Inclusive-exclusive principle:** $P(A \bigcup B) = P(A) + P(B) - P(A \bigcap B)$

    $P(A \bigcup B \bigcup C) = P(A) + P(B) + P(C) - P(A \bigcap B) - P(A \bigcap C) - P(B \bigcap C) - P(A \bigcap B \bigcap C)$
11. **DeMorgan's rule:**

    $A' \bigcap B' = (A \bigcup B)'$

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12. **Venn Diagram**

Section 2.2

List of topics:

1. **Binomial coefficients (will study in detail in 2.4. Just need to know how to compute here.)**
2. **Conditional Probability:** $P(A|B) = \dfrac{P(A \bigcap B)}{P(B)}$
3. **Product rule:**
   $P(A \bigcap B) = P(A|B)P(B) = P(B|A)P(A)$
   $P(A \bigcap B \bigcap C) = P(A)P(B|A)P(C|A \bigcap B)$
4. **Independent: Two events are** *independent* **if**
   $P(A \bigcap B) = P(A)P(B)$
   which means $P(A|B) = P(A)$ **and** $P(B|A) = P(B)$
5. **Tree Diagram**
6. **Partition (Pizza): If**
   1) $A_1, A_2, ..., A_k$ **are mutually exclusive,**
   2) $\bigcup_{i=1}^{k} A_i = S$**;**
   3) $P(A_i) > 0$ **for all** $i = 1, 2, ..., k.$
   Then we say that $A_1, A_2, ..., A_k$ **form a partition.**
7. **The Law of Total Probability: If** $A_1, A_2, ..., A_k$ **form a partition, then for any event** B**,**
   $P(B) = \Sigma_{j=1}^{k} P(B|A_j)P(A_j)$
8. **Smallest partition:** $S = A \bigcup A'$
   $P(B) = P(B|A)P(A) + P(B|A')P(A')$
9. **Bayes' Theorem: If** $A_1, A_2, ..., A_k$ **form a partition, then for any event** B**,**
   $P(A_j|B) = \dfrac{P(B|A_j)P(A_j)}{P(B)}$
10. **Independence vs. Mutually exclusive**

Sections 2.3

List of topics:

1. **Random variable(r.v.): a** *random variable* **is a function from sample space to real numbers.**
2. **Probability distribution function for discrete r.v.(pdf)**

   1) $P(X = k) \geq 0$;

   2) $\Sigma_k P(X = k) = 1$.
3. **How to denote pdf?**

   By table;

   By formula;

   Histogram;

   Relative frequency table.
4. **Expectation of a discrete random variable:**

   $\mu = E(X) = \Sigma_{\textbf{all possible } k} k \cdot f(k)$
5. **Expectation of a function of a discrete random variable:**

   $Eu(X) = \Sigma_{\textbf{all possible } k} u(k) \cdot f(k)$
6. **Expectation is linear:** $E(aX + bY) = aE(X) + bE(Y)$ **for any constants** a, b**, and any random variables** X, Y.
7. **Variance of X:**

   $\sigma^2 = V(X) = E(X - \mu)^2$ **(Good for understanding)**

   $\sigma^2 = V(X) = E(X^2) - \mu^2 = (\Sigma k^2 \cdot f(k)) - \mu^2$ **(Good for computation)**
8. **Variance is not linear.**

   i.e. $V(aX + bY) \neq aV(X) + bV(Y)$

   Actually $V(aX + bY) = a^2V(X) + b^2V(Y) + 2abCov(X, Y)$
9. **Standard deviation of** X**:** $\sigma = \sqrt{V(X)}$
10. **Chebychev inequality:**

    $P(\mu - k\sigma \leq X \leq \mu + k\sigma) \geq 1 - \frac{1}{k^2}$

Sections 2.4

List of topics: Binomial Distribution

1. **Binomial expansion:** $(x+y)^n$
2. **Binomial coefficients:** $C(n,k)$

   1) By Yang's triangle

   2) By combination: $C(n,k) = \dfrac{P(n,k)}{k!} = \dfrac{n!}{k!(n-k)!}$
3. **Bernoulli trials:**
4. **Binomial distribution: check 3 conditions: exactly 2 outcomes for each trial; independent trials; probability of head stays the same for all trials.**

   Model: Toss a coin n times.

   $P(H) = p, P(T) = q, q = 1 - p$.

   X= number of heads.
5. **p.d.f.** $P(X=k) = C(n,k)p^k q^{n-k},\ k = 0, 1, 2, ..., n$
6. **Expected value** $E(X) = np$; **or** $\mu = np$

   Variance $V(X) = npq$; **or** $\sigma^2 = npq$

   Standard deviation $\sigma = \sqrt{npq}$
7. **Negative Binomial distribution: Toss a coin until rth heads**
8. **Hypergeometric distribution: without replacement**

Sections 2.5

List of topics: Poisson Distribution

1. **Model: Counting number of occurences**
2. **pdf:** $P(X = k) = \dfrac{e^{-\lambda}\lambda^k}{k!},\ k = 0, 1, 2, ...$
3. **mean and variance are both** λ
4. **Closely related to Exponential distribution. Will discuss in section 3.3.**

Section 2.6 Dist of 2 discrete random variables

List of topics:

1. **CDF of X_1 and X_2:**

   $F_{X_1,X_2}(x,y) = P[(X_1 \leq x) \bigcap (X_2 \leq y)]$

2. **Discrete case: Joint pdf:**

   $f_{X_1,X_2}(x,y) = P(X_1 = x, X_2 = y)$

   Note: This f is a probability, hence $0 \leq f \leq 1$

3. **Marginal pdfs**

4. **Expectations**

5. $E(u(X_1, X_2))$

6. **Conditional distribution and expectations**

7. **Conditional pdf is the ratio of joint pdf to marginal pdf.**

   $p_{X|Y}(x|y) = \dfrac{p_{X,Y}(x,y)}{p_Y(y)}$ **for y in the support of Y.**

   $p_{Y|X}(y|x)$ **is Similarly defined.**

8. **Conditional expectation:** $E(u(Y)|x) = \int_{-\infty}^{\infty} u(y) f_{Y|X}(y|x)dy$ **(which is a function of x.)**

   Hence $E(Y|X)$ is a random variable of X.

9. **Covariance of X and Y:**

   $cov(X,Y) = E[(X - \mu_X)(Y - \mu_Y)]$ **(easier to understand)**

   $cov(X,Y) = E(XY) - \mu_X \mu_Y$ **(easier for calculation)**

10. **Correlation coefficient** $\rho = \dfrac{cov(X,Y)}{\sigma_X \sigma_Y}$

11. **Equivalent conditions: X and Y are independent iff**

    (1) By pdf: $f_{X,Y}(x,y) == f_X(x) f_Y(y)$ **(note the regions.)**

    (2) By CDF: $F(x,y) = F_X(x) F_Y(y)$ **for all** (x,y)

    (3) By MGF: $M(t_1,t_2) = M(t_1,0)M(0,t_2)$

12. **How can you tell two r.v. X and Y are independent by joint density?**

    Region and function

13. What happens to covariance(and corelation coefficient) if X and Y are independent?

If X and Y are independent, then $Cov(X,Y) = 0$, hence $\rho = 0$.

However, if $Cov(X,Y) = 0$, or $\rho = 0$, X and Y may not be independent.

Facts:

- $-1 \leq \rho \leq 1$
- $\rho = \pm 1$ **implies"perfect linear relationship between X and Y."**
- $\rho = 0$ **implies "No linear relationship between X and Y (but can be quadratic, for example)."**
- $\rho > 0$ **means X and Y are positively related (If X increases, so is Y).**
- $\rho < 0$ **means X and Y are negatively related (If X increases, Y decreases).**
- **If X and Y are independent, then $Cov(X,Y) = 0$, hence $\rho = 0$.**

  However, if $Cov(X,Y) = 0$, or $\rho = 0$, X and Y may not be independent.

  Ex: 2.5.10.
- **Special case: If X and Y are normal (covered later), then $Cov(X,Y) = 0$, or $\rho = 0$ implies "X and Y are independent."**