# STAT 401 Review for Exam 2 Solutions Lia Liu

**Topics:** You are allowed to take one page of formula sheet(regular 8.5 in by 11 in) to the midterm 2.

1. Joint pdf/pmf, marginal pdf/pmf, conditional pdf/pmf.
2. How to compute probability, MGF, $E(X), E(XY), E(X|y), V(X), Cov(X < Y), \rho$ given joint pdf/pmf?
3. Independence, $E(\Sigma_{i=1}^{n} a_i X_i)$, in particular, if $a_i = 1$ or $a_1 = 1/n$. Compute $V(\Sigma_{i=1}^{n} a_i X_i)$. What if $X_i$'s are independent? What is $V(\Sigma_{i=1}^{n} a_i X_i)$ if $X_i$'s are iid with mean $\mu$ and variance $\sigma^2$? In particular, if $a_i = 1$ or $a_1 = 1/n$.
4. Transformation
5. Special distributions: Especially Binomial, Geometric, Exponential, Normal.
6. Taylor expansion of $e^x$, $\frac{1}{1-x}$, and its term-by-term differentiation.

**Problems:**

1. Suppose we draw 2 balls out of an urn with 8 red, 6 blue, and 4 green balls. Let $X$ be the number of red balls we get and $Y$ the number of blue balls.

   (a) Find the joint pmf of $X$ and $Y$.

| $Y$ | $X = 0$ | 1 | 2 | $f_Y$ |
|---|---|---|---|---|
| 0 | 6/153 | 32/153 | 28/153 | 66/153 |
| 1 | 24/153 | 48/153 | 0 | 72/153 |
| 2 | 15/153 | 0 | 0 | 15/153 |
| $f_X$ | 45/153 | 80/153 | 28/153 | |

   (b) Find the marginals of $X$ and $Y$.

   (c) Find the conditional pmf of $f_{Y|x}(y|x = 1)$

| $Y|X = 1$ | 0 | 1 | 2 |
|---|---|---|---|
| $P(Y|X = 1)$ | 32/80 | 48/80 | 0 |

   (d) Find $P(Y = 1|X = 1) = 48/80$

2. Suppose $P(X = x, Y = y) = c(x + y)$ for $x, y = 0, 1, 2, 3$.

   (a) What value of $c$ will make this a joint pmf? $c = 1/48$

   (b) What is $P(X > Y)$? 3/8

   (c) Find the moment generating function of $X$ and $Y$. (Messy)

3. Suppose $X$ and $Y$ has joint density $f(x, y) = c(x + y)$ for $0 < x, y < 1$.

   (a) What is $c$? $c = 1$

   (b) What is $P(X < 1/2)$? 3/8

   (c) What is $P(X + Y > 1/2)$? 23/24

4. Suppose $X$ and $Y$ has joint density $f(x, y) = 2$ for $0 < x < y < 1$.

   (a) Find $P(Y - X > z) = (z-1)^2$, for $0 \le z \le 1$; $= 0$ otherwise.

   (b) Find the marginal densities of $X$ and $Y$. $f_X(x) = 2(1-x),\ 0 < x < 1$; $f_Y(y) = 2y,\ 0 < y < 1$;

   (c) $M(t_1, t_2) = (2/t_1)(\frac{e^{t_1+t_2}}{t_1+t_2} - \frac{e^{t_2}}{t_2} - \frac{1}{t_1+t_2} + \frac{1}{t_2})$

   (d) Find $E(X), E(Y), V(X), V(Y), Cov(X, Y)$

   $E(X) = 1/3,\ E(Y) = 2/3,\ V(X) = V(Y) = 1/18,\ cov(X, Y) = 1/36$

5. Suppose $X$ and $Y$ has joint density $f(x, y) = 6xy^2$ for $0 < x, y < 1$. What is $P(X + Y < 1)$? 1/10

6. Suppose $X, Y$, and $Z$ have uniform density on the unit cube. Find $P(X+Y+Z < 1)$. which is the volume of the pyramid $= 1/6$.

7. Suppose $X_1$ and $X_2$ are independent and have uniform density on $(0, 1)$. Let $Y = X_1/X_2$, and $Z = X_1X_2$.

   (a) Find the joint density of $Y$ and $Z$.

   The joint density of $X_1, X_2$ is $f(x_1, x_2) = 1,\ 0 < x_1 < 1, 0 < x_2 < 1$.

   Solve the old variables in terms of new: $x_1 = \sqrt{yz}; x_2 = \sqrt{z/y}$

   The Jacobian then is $J = \begin{vmatrix} \frac{1}{2}z^{\frac{1}{2}}y^{-\frac{1}{2}} & \frac{1}{2}z^{-\frac{1}{2}}y^{\frac{1}{2}} \\ -\frac{1}{2}z^{\frac{1}{2}}y^{-\frac{3}{2}} & \frac{1}{2}z^{-\frac{1}{2}}y^{-\frac{1}{2}} \end{vmatrix} = 1/(2y)$.

   $0 < x_1 < 1$ implies that $0 < \sqrt{yz} < 1$, square it, $0 < yz < 1$, or $z < 1/y$; $0 < x_2 < 1$ implies $0 < \sqrt{z/y} < 1$, square it, $0 < z/y < 1$, so $z < y$;

   Since $x_i$ 's are positive, so are $y$ and $z$.

   Now draw $z = 1/y, z = y$ and pick points to choose the region for the inequalities.

   $f_{Y,Z}(y, z) = 1/(2y),\ 0 < yz < 1, 0 < z < y, 0 < z < 1$ (Note: the region is bounded by $y = z, z = 1/y$ and y axis.

   (b) Find the marginals of $Y$ and $Z$.

   $$f_Y(y) = \begin{cases} \frac{1}{2} & \text{if } 0 < y < 1 \\ \frac{1}{2y^2} & \text{if } 1 \le y < \infty \end{cases}$$

   $f_Z(z) = -\ln z,\ 0 < z < 1$

8. Using the clues (a) $P(Y = 2|X = 0) = 1/4$; (b) $X$ and $Y$ are independent, fill in the rest of the joint pmf.

   Step 1: By independence, $P(Y = 2|X = 0) = P(Y = 2) = 0.25$, hence $P(X = 6, Y = 2) = 0.25 - 0.1 - 0.05 = 0.1$. $P(Y = 1) = 1 - 0.25 = 0.75$

   Step 2: Let $P(X = 0) = a$, then $P(X = 0, Y = 1) = P(X = 0)P(Y = 1) = 0.75a$, look at the 1st column, $0.75a + 0.1 = a$ implies that $a = 0.4$; So $P(X = 0, Y = 1) = P(X = 0)P(Y = 1) = 0.75a = 0.3$;

Similarly, let $P(X=3)=b$, then $P(X=3,Y=1)=P(X=3)P(Y=1)=0.75b$, look at the 2nd column, $0.75b+0.05=b$ implies that $b=0.2$; so $P(X=3,Y=1)=0.15$;

let $P(X=6)=c$, then $P(X=6,Y=1)=P(X=6)P(Y=1)=0.75c$, look at the 3rd column, $0.75c+0.1=c$ implies that $c=0.4$; so $P(X=6,Y=1)=0.3$;

The joint pmf is

| Y | X=0 | 3 | 6 |
|---|---|---|---|
| 1 | 0.3 | 0.15 | 0.3 |
| 2 | 0.1 | 0.05 | 0.1 |

9. Two people have agreed to meet at a party. If we call the start of the party time 0, then Mary (who wants to be late but not too late) arrives at time $X$ (measured in hours) with density $f_X(x)=1$, for $0<x<1$, while John (the space cadet) arrives at time $Y$ with density $f_Y(y)=e^{-y}$ for $y>0$. Suppose that the arrival times $X$ and $Y$ are independent, and Mary will get impatient and leave if John arrives more than one hour after she does. What is the probability Mary will leave before John arrives, i.e. what is $P(Y>X+1)$? $e^{-1}-e^{-2}$

   Joint density is $f(x,y)=e^{-y},\ 0<x<1, y>0$,

   $P(Y>X+1)=\int_{x=0}^{1}\int_{y=x+1}^{\infty}e^{-y}dydx=e^{-1}-e^{-2}$

10. If $X_i,\ i=1,2,...,9$ are iid's with mean $\mu=-5$ and standard deviation $\sigma=2$.

    (a)the mean and variance of $\frac{1}{9}\Sigma_{i=1}^{9}X_i$? mean $-5$, variance $4/9$

    (b) the mean and variance of $\Sigma_{i=1}^{9}X_i$? mean $-45$, variance 36.

11. If $X_i,\ i=1,2,...,25$ are iid's Bernoulli trials $B(1,p)$, with $p=1/4$.

    (a) What is the mean and variance of $\hat{p}=\frac{1}{25}\Sigma_{i=1}^{25}X_i$? mean $1/4$, variance $3/400$

    (b) What is the mean and variance of $X=\Sigma_{i=1}^{25}X_i$? What is the name of this distribution?

    mean $25/4$, variance $75/16$. Binomial with $n=25, p=1/4$

12. If $X$ and $Y$ are iid exponential distribution with mean $1/\lambda$, which means the density of $X$ is $f(x)=\lambda e^{-\lambda x},\ x>0$

    (a) Find the density function of $X+Y$.

    Let $X=X, Z=X+Y$, then $X=X, Y=Z-X$, The Jacobian $J=1$. The joint density of $X,Y$ is $f(x,y)=\lambda^2e^{-\lambda x}e^{-\lambda y}$. Therefore the joint density of $X,Z$ is $f(x,z)=\lambda^2e^{-\lambda z}\cdot 1$, $x>0, y=z-x>0$ implies the domain is $x>0, z>x>0$

    The marginal for $Z$ is $f_Z(z)=\int_0^z\lambda^2e^{-\lambda z}dx=\lambda^2ze^{-\lambda z},\ z>0$

    Note: This is not Exponential anymore. Exponential distribution is not stable. Can you name this distribution?

    (b) If $\lambda=1$, find $P(X>Y>2)=e^{-2}-\frac{e^{-4}}{2}$

13. Let $X$ be the number of customers login on a web site per minute. Assume $X$ has a Poisson distribution with a mean of 6 login requests per minute.

(a) What is the probability that no one requested to log on this site in the next minute? $P(X = 0) = e^{-6}$

(b) Let $W$ be the time in minutes between the 2nd and 3rd requests. What is the distribution name of $W$? What is the expected value of $W$? Exponential with mean 1/6 minute.

14. If a r.v $X$ satisfies a normal distribution with mean 60 and standard deviation 20. Find the following results using the Empirical rule (68-95-99.7 rule):

(a) $P(X \geq 80) = 16\%$

(b) $P(|X - 60| < 40) = 95\%$

(c) $P(|X - 60| \geq 20) = 32\%$

(d) $P(X < 20) = 2.5\%$

15. If a r.v. $X$ satisfies a normal distribution with mean 2, variance 9 and a r.v. $Y$ satisfies a normal distribution with mean 3, variance 4. Suppose $X$ and $Y$ are independent.

(a) What is the mean and variance of $X + Y$? $E(X + Y) = E(X) + E(Y) = 5$, and $V(X + Y) = V(X) + V(Y) = 13$

(b) What is the mean and variance of $X - Y$? $E(X - Y) = E(X) - E(Y) = -1$, and $V(X - Y) = V(X) + V(Y) = 13$

(c) What is the mean and variance of $2X - 3Y$? $E(2X - 3Y) = 2E(X) - 3E(Y) = -5$, and $V(2X - 3Y) = 4V(X) + 9V(Y) = 72$

16. If a r.v.$X$ satisfies a Poisson distribution with mean 2 and a r.v. $Y$ satisfies a Poison distribution with mean 3. Suppose $X$ and $Y$ are independent.

(a) What are the variances of $X$ and $Y$? $V(X) = 2, V(Y) = 3$

(b) What is the mean and variance of $X + Y$? $E(X + Y) = 5, V(X + Y) = 5$